Towards a stable trace formula for metaplectic groups and its applications

Wen-Wei Li

Institut de Mathématiques de Jussieu

June 30, 2011
BIRS Workshop on L-packets

References

1. *Transfert d'intégrales orbitales pour le groupe métaplectique* (Compositio Math. 147 / arXiv)
2. *Le lemme fondamental pondéré pour le groupe métaplectique* (arXiv)
3. *La formule des traces pour les revêtements de groupes réductifs connexes. I.*
   Le développement géométrique fin (arXiv)
4. *La formule des traces pour les revêtements de groupes réductifs connexes. II.*
   Analyse harmonique locale (in preparation)
5. *La formule des traces pour les revêtements de groupes réductifs connexes. III.*
   Le développement spectral fin (in preparation)

The local case

F: local field of characteristic $\neq 2$. $\psi : F \to \mathbb{C}^\times$ nontrivial unitary character.

- W: finite-dimensional F-vector space,
- $\langle , \rangle$: symplectic form on W.

Let $\mathrm{Sp}(W)$ be the symplectic group. If $\dim W = 2n$, we sometimes write $\mathrm{Sp}(2n)$ since $(W, \langle , \rangle)$ is unique up to isomorphism.

- We can construct the metaplectic covering, which is a central extension
$$1 \to \{1, \epsilon\} \to \widetilde{\mathrm{Sp}}(W) \xrightarrow{\mathbf{p}} \mathrm{Sp}(W) \to 1.$$
The covering $\mathbf{p}$ is nontrivial and nonalgebraic (i.e. does not come from an isogeny of algebraic groups) when $F \neq \mathbf{C}$. $\widetilde{\mathrm{Sp}}(W)$ will also be denoted by $\widetilde{\mathrm{Sp}}(2n)$, $\widetilde{\mathrm{Sp}}(2n, F)$, $\widetilde{\mathrm{Sp}}(W, F)$.
- The Weil representation of $\widetilde{\mathrm{Sp}}(W)$: $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$, where $\omega_\psi^\pm$ are unitary irreps.
- If res.char$(F) > 2$ and $L \subset W$ is a self-dual lattice w.r.t. $\psi \circ \langle , \rangle$, then $K := \mathrm{Stab}(L)$ is hyperspecial and $\mathbf{p}$ is canonically split over K.

Genuine representations

- It suffices to study the representations π of $\widetilde{\mathrm{Sp}}(W)$ which are *genuine*, i.e. $\pi(\epsilon) = -\mathrm{id}$. Examples: $\omega_\psi^\pm$.
- Test functions: it suffices to consider $\pi(f)$ with *antigenuine* $f \in C_c^\infty(\widetilde{\mathrm{Sp}}(W))$, i.e. $f(\epsilon\cdot) = -f(\cdot)$.

The adélic case

F a number field, $\mathbb{A}$ its ring of adèles, $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ a nontrivial unitary character.

- Construct the adélic $\widetilde{Sp}(W, \mathbb{A})$ and ω_ψ as before.
- The same notions of genuine/antigenuine objects.
- We can show that

$$\widetilde{Sp}(W, \mathbb{A}) = \left(\prod_v {}' \widetilde{Sp}(W, F_v) \right) / \mathbf{N},$$

$$\mathbf{N} := \left\{ (\epsilon_v)_v \in \bigoplus_v \{\pm 1\} : \prod_v \epsilon_v = 1 \right\}.$$

Some problems

- Study of genuine representations on $\widetilde{Sp}(W)$; Howe correspondence.
- Endoscopy of the nonalgebraic group $\widetilde{Sp}(W)$?
- Invariant trace formula for $\widetilde{Sp}(W, \mathbb{A})$?
- Stable trace formula for $\widetilde{Sp}(W, \mathbb{A})$?

It's possible to pursue the invariant trace formula for more general coverings of connected reductive algebraic groups. Cf. [Mezo].

Coverings of connected reductive groups

F: local field, G: connected reductive F-group.

- $G = \mathrm{Sp}(2n)$: A. Weil (1964) $\Rightarrow$ representation-theoretic interpretation of Siegel modular forms of half-integral weight.
- $G = \mathrm{SL}(2)$ or $\mathrm{GL}(2)$: Shimura (1973), Kubota.
- G split, simple and simply connected: R. Steinberg (1962), H. Matsumoto (1969) constructed the *universal central extension* of $G(F)$. Related to algebraic K-theory.
- $G = \mathrm{GL}_n$: metaplectic correspondence (Flicker, Kazhdan, Patterson,..., ≥ 1980).
- G arbitrary: Deligne and Brylinski (2001) classified their $\mathbf{K}_2$-extensions.

There are also adélic constructions.

From the viewpoint of trace formula...

Consider the most general case of topological central extensions

$$1 \to \mathbf{N} \to \tilde{G} \to G(k) \to 1$$

where

- F: global or local field of char 0,
- G: connected reductive over F,
- $k = F$ (local) or $k = \mathbb{A}$ (global),
- $\mathbf{N}$: finite abelian group.

We may assume $\mathbf{N} = \mu_m := \{z \in \mathbb{C}^\times : z^m = 1\}$.
Genuine representations: $\pi(\epsilon) = \epsilon \cdot \mathrm{id}$, $\forall \epsilon \in \mu_m$.
Antigenuine functions: $f(\epsilon \cdot) = \epsilon^{-1} f(\cdot)$, $\forall \epsilon \in \mu_m$.

The class of such extensions should be:

- stable under push-forward $\mu_m \to \mu_{m'}$;
- stable under passage to Levi subgroups (*philosophy of cusp forms*);
- when F is global,
  - $\exists$ splitting over $G(F)$ ($\Rightarrow$ spectral decomposition, see [MW]),
  - $\exists$ splittings over hyperspecial subgroups $G(\mathfrak{o}_v)$ at almost all v, here we fix an integral model of G;
  - (continued) the corresponding spherical Hecke algebras must be commutative ($\Rightarrow \bigotimes$-decomposition of admissible irreps)

Good splittings over unipotent subgroups: automatic. $\Rightarrow$ notions of constant terms and Jacquet functors.

These conditions are satisfied by the $\mathbf{K}_2$-extensions of Brylinski-Deligne.

Statement

$\mathbf{p} : \tilde{G} \to G(\mathbb{A})$ be a covering, $\mathrm{Ker}(\mathbf{p}) = \mu_m$.

- $\tilde{G}^1 := \mathrm{Ker}(H_G \circ \mathbf{p})$ where $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$ is the Harish-Chandra homomorphism.
- R: right regular representation of $\tilde{G}$ on $L^2(G(F)\backslash\tilde{G}^1)$,
- $f \in C_c^\infty(\tilde{G})$ antigenuine,
- K the kernel of $R(f)$, $k(x) := K(\tilde{x}, \tilde{x})$ for $x \in G(\mathbb{A})$, $\tilde{x} \in \mathbf{p}^{-1}(x)$,
- for any parabolic $P = MU$, R_P the right regular representation on $L^2(U(\mathbb{A})M(F)\backslash\tilde{G}^1)$ and K_P its kernel.

Fix minimal Levi M_0 and maximal compact $K \subset G(\mathbb{A})$ in good relative position. Set $\tilde{K} := \mathbf{p}^{-1}(K)$.

Fix $P_0 \in \mathcal{P}(M_0)$. For $T \in \mathfrak{a}_0$, define the truncated kernel à la Arthur

$$k^T(x) := \sum_{P \supset P_0} (-1)^{\dim A_P/A_G} \sum_{\delta \in P(F)\backslash G(F)} K_P(\delta\tilde{x}, \delta\tilde{x}) \hat{\tau}_P(H_P(\delta x) - T).$$

Theorem

For T highly regular, $k^T(x)$ is integrable over $G(F)\backslash \tilde{G}^1$. There is an identity of absolutely convergent integrals

$$\sum_{\mathfrak{o}} J^T_{\mathfrak{o}}(f) = J^T(f) = \sum_{\chi} J^T_{\chi}(f).$$

Everything in sight is polynomial in T.

- Spectral side: χ ranges over *cuspidal data* (M, σ), where $M \supset M_0$ is a Levi subgroup, σ is a cuspidal automorphic representation of $\tilde{M}$ inside $L^2(M(F)\backslash\tilde{M}^1)$.
- Geometric side: $\mathfrak{o}$ ranges over semisimple classes in $G(F)$. The unipotent term $J^T_{\mathrm{unip}}(f)$ corresponds to 1.

About the proof

- Combinatorics: the same as the case of reductive groups (Arthur),
- Spectral decomposition: included in [MW],
- Geometric side: the same as in the case of reductive groups – we only look at conjugacy classes in $G(F)$.

The problem of refinement

Let $T_0 \in \mathfrak{a}_0$ be the canonical element (depending on K) defined by Arthur. Then $J(f) := J^{T_0}(f)$, $J_\chi(f) := J_\chi^{T_0}(f)$, $J_\mathfrak{o}(f) := J_\mathfrak{o}^{T_0}(f)$ are independent of P_0. The problem is to find explicit formulas for them.

Desiderata

Express $J_\chi(f)$, $J_\mathfrak{o}(f)$ in terms of weighted orbital integrals and weighted characters (local objects), with global coefficients.

Descend to the unipotent case

Idea: get rid of the covering.
For $x, y \in G(\mathbb{A})$ with liftings $\tilde{x}, \tilde{y} \in \tilde{G}$, set

$$[x, y] := \tilde{x}^{-1}\tilde{y}^{-1}\tilde{x}\tilde{y}.$$

Let $\sigma \in G(F)$ be semisimple, $G_\sigma := Z_G(\sigma)^\circ$. Then $[\cdot, \sigma]$ defines a homomorphism $G_\sigma(\mathbb{A}) \to \mu_m$.

Principle

Let $\mathfrak{o}$ be the $G(F)$-orbit containing σ. Reduce $J_{\mathfrak{o}}^{\tilde{G}}(f)$ to $J_{\text{unip}}^{G_\sigma, [\cdot,\sigma]}$, the unipotent term of the trace formula of G_σ twisted by the character $[\cdot, \sigma]$. (More precisely, one must consider some Levi subgroups of G_σ).

Remark: we have Jordan decomposition on coverings.

Example: descent of orbital integrals

The same formalism about $[\cdot, \sigma]$ applies to the local case. Let F be a local field, $\mathbf{p} : \tilde{G} \to G(F)$ a covering and $f \in C_c^\infty(\tilde{G})$ antigenuine.

Theorem

$\exists f^\flat \in C_c^\infty(\mathfrak{g}_\sigma(F))$ such that $\forall \tilde{\gamma} = \sigma \exp(X)$ with $X \in \mathfrak{g}_\sigma(F)$ sufficiently close to 0, we have

$$|D^G(\gamma)|^{\frac{1}{2}} O_{\tilde{\gamma}}^{\tilde{G}}(f) = |D^{G_\sigma}(X)|^{\frac{1}{2}} O_X^{G_\sigma, [\cdot,\sigma]}(f^\flat)$$

where D^G and D^{G_σ} are the Weyl discriminants on G and $\mathfrak{g}_\sigma$, respectively.

It's tempting to remove $[\cdot, \sigma]$ and replace $G_\sigma(F)$ by $\mathrm{Ker}([\cdot, \sigma])|_{G_\sigma(F)}$. However the latter group is less manageable.

Basic ideas for the refined geometric expansion

1. Reduce to the study of $J_{\mathrm{unip}}^{G_\sigma,[\cdot,\sigma]}(f)$.
2. Express $J_{\mathrm{unip}}^{G_\sigma,[\cdot,\sigma]}(f)$ in terms of weighted unipotent orbital integrals twisted by $[\cdot,\sigma]$ (need to adapt Arthur's arguments).
3. Define weighted orbital integrals on $\tilde{G}$ and deduce analogous descent formulas.
4. Compare the descent formulas to express $\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f)$ in terms of weighted orbital integrals on $\tilde{G}$.

Result: expressed in terms of *good weighted orbital integrals* $J_{\tilde{M}_S}(\tilde{\gamma}_S, f_S)$, where

- S: a finite set of places containing the archimedean ones, depending on $\mathrm{Supp}(f)$;
- $\tilde{M}_S := \mathbf{p}^{-1}(M(F_S))$;
- $f = f_S f_{K^S}$, where f_{K^S} is the unit in the antigenuine spherical Hecke algebra;
- $\tilde{\gamma}_S$: conjugacy class in $\tilde{M}_S$ which is *good*, i.e. $\tilde{x}\tilde{\gamma}_S = \tilde{\gamma}_S\tilde{x}$ iff $x\gamma_S = \gamma_S x$, where $\mathbf{p}(\tilde{*}) = *$.

The definition of weighted orbital integrals is analogous to Arthur's: Langlands' geometric argument is applied after descent, thus the covering creates no difficulty.

We have:

$$J(f) = \sum_{M \in \mathcal{L}(M_0)} \frac{|W_0^M|}{|W_0^G|} \sum_{\substack{\gamma \in (M(F))_{M,S}^{K,\mathrm{good}} \\ \gamma \rightsquigarrow \widetilde{\gamma_S}}} a^{\tilde{M}}(S, \dot{\widetilde{\gamma_S}}) J_{\tilde{M}_S}(\dot{\widetilde{\gamma_S}}, f).$$

- S: depending on $\mathrm{Supp}(f)$.
- $(M(F))_{M,S}$: the set of (M,S)-equivalence classes defined by Arthur.
- $(\cdots)_{M,S}^{K,\mathrm{good}}$: the classes γ admitting a representative whose components outside S are in K^S, and whose component $\widetilde{\gamma_S}$ in $\tilde{M}_S$ is *good* (using $\mathbf{p}^{-1}(M(F_S) \times K^S) = \tilde{M}_S \times K^S$).
- The correspondence $\gamma \rightsquigarrow \widetilde{\gamma_S}$: described as above.

One difficulty

- In the course of proof, one must show that $a^{\tilde{M}}(S, \dot{\widetilde{\gamma_S}}) J_{\tilde{M}_S}(\dot{\widetilde{\gamma_S}}, f)$ behaves well with respect to the correspondence $\gamma \rightsquigarrow \widetilde{\gamma_S}$.
- Unlike the case treated by Arthur, the character $[\cdot, \sigma]$ intervenes and we need a somewhat technical result of "transport of structure" for $a^{\tilde{M}}(S, \dot{\widetilde{\gamma_S}})$ and $J_{\tilde{M}_S}(\dot{\widetilde{\gamma_S}}, f)$.

The refined spectral side

Suppose f to be $\tilde{K}$-finite from now on. Then

$$J_\chi(f) = \sum_{M \in \mathcal{L}(M_0)} \sum_{\pi \in \Pi(\tilde{M}^1)} \sum_{L \in \mathcal{L}(M)} \sum_{s \in W^L(M)_{\mathrm{reg}}} \frac{|W_0^M|}{|W_0^G|} \cdot$$

$$\cdot \, |\det(s-1|\mathfrak{a}_M^L)|^{-1} \int_{i(\mathfrak{a}_L^G)^*} \mathrm{tr}(\mathcal{M}_L(\tilde{P},\lambda) M_{P|P}(s,0) \mathcal{I}_{\tilde{P}}(\lambda, f)_{\chi,\pi}) \mathrm{d}\lambda.$$

- $\Pi(\tilde{M}^1)$: the set of unitary irreps of $\tilde{M}^1$ up to equivalence,
- $P \in \mathcal{P}(M)$ arbitrary,
- $M_{P|P}(s,0)$: global intertwining operators,
- $\mathcal{M}_L(\tilde{P},\lambda)$: an operator defined by a (G,L)-family arising from intertwining operators,
- $\mathcal{I}_{\tilde{P}}(\cdots)$: unitary parabolic induction.

Local ingredients of the proof

Mostly Harish-Chandra's theory:

- local intertwining operators,
- c-functions, μ-functions,
- Plancherel formula,
- normalization of intertwining operators:
  - Archimedean case: juggling with Γ-functions,
  - non-archimedean case: adapt Langlands' proof,
  - "unramified" case: need some theory of unramified genuine principal series, cf. [McNamara].

Global ingredients of the proof

In view of [MW], one can simply copy Arthur's arguments.

The next steps

1. Define and study the weighted characters for coverings.
2. Express the spectral side in terms of weighted characters.
3. The invariant trace formula: need the trace Paley-Wiener theorem for $\tilde{K}_\infty$-finite functions?
4. To complete the induction step defining the invariant trace formula, Arthur used a generalization of Kazhdan's "Theorem 0" proved using global arguments, which becomes problematic even for the coverings constructed by Steinberg (weak approximation of good elements might fail). $\Leftarrow$ Bypass this problem by establishing the invariant local trace formula for coverings!

Remark: No difficulty in the case $\tilde{G} = \widetilde{\mathrm{Sp}}(W)$!

Let's come back to $\mathbf{p} : \widetilde{\mathrm{Sp}}(W) \to \mathrm{Sp}(W)$ with $\dim_F W = 2n$, and F: local field of characteristic 0. Let $\mathrm{SO}(2n+1)$ denote the *split* odd orthogonal group of rank n.
Idea: the genuine representation theory of $\widetilde{\mathrm{Sp}}(W)$ is closely related to $\mathrm{SO}(2n+1)$.

Evidences

- Prior works on Howe correspondence for the pair $(\mathrm{Sp}(2n), \mathrm{O}(V, q))$ where $\dim V = 2n+1$.
- Savin's work on the antigenuine spherical Hecke algebra of $\widetilde{\mathrm{Sp}}(W)$ when $\mathrm{char}(F) > 2$: it's isomorphic to the Hecke algebra of $\mathrm{SO}(2n+1)$.

The dual group

Set $G := \mathrm{Sp}(W)$ and $\tilde{G} := \widetilde{\mathrm{Sp}}(W)$. Define

$$\hat{\tilde{G}} := \mathrm{Sp}(2n, \mathbb{C})$$

with trivial Galois action. That is, $\tilde{G}$ and $\mathrm{SO}(2n+1)$ have the same dual group. But what's the difference?

Rule

We should replace $Z_{\hat{\tilde{G}}} = Z_{\hat{\tilde{G}}}^{\Gamma}$ by its identity component, i.e. the trivial group.

As a result, the endoscopy groups of $\tilde{G}$ are different from those of $\mathrm{SO}(2n+1)$. The study of weighted fundamental lemma will give further evidences.

Elliptic endoscopic data

Given the definitions above, one can imitate Langlands' definition to get the elliptic endoscopic data. However it's more convenient to define them directly.

Definition

The elliptic endoscopic data of $\tilde{G}$ are the pairs (n', n'') with $n', n'' \in \mathbb{Z}_{\geq 0}$ such that $n' + n'' = n$. To a pair (n', n''), the associated endoscopic group is

$$H := \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1).$$

Remark A. Unlike the case of $\mathrm{SO}(2n+1)$, there is no symmetry $(n', n'') \leftrightarrow (n'', n')$.

Remark B. Renard's formalism: $H := \widetilde{\mathrm{Sp}}(2n') \times \widetilde{\mathrm{Sp}}(2n'')$.

Correspondence of semisimple conjugacy classes

Fix (n', n'') and H as above. Let $\gamma = (\gamma', \gamma'') \in H(F)$ be semisimple such that γ' (resp. γ'') as an element in $\mathrm{GL}(2n'+1)$ (resp. $\mathrm{GL}(2n''+1)$) has eigenvalues

$$a'_1, a'_2, \ldots, a'_{n'}, 1, {a'_{n'}}^{-1}, \ldots, {a'_1}^{-1}$$

.

$$(\text{resp. } a''_1, a''_2, \ldots, a''_{n''}, 1, {a''_{n''}}^{-1}, \ldots, {a''_1}^{-1}).$$

We say $\delta \in G(F)$ corresponds to γ, written $\gamma \leftrightarrow \delta$, if δ is semisimple with eigenvalues

$$\begin{gathered} a'_1, a'_2, \ldots, a'_{n'}, {a'_{n'}}^{-1}, \ldots, {a'_1}^{-1}, \\ -a''_1, -a''_2, \ldots, -a''_{n''}, -{a''_{n''}}^{-1}, \ldots, -{a''_1}^{-1}. \end{gathered}$$

This defines a correspondence of semisimple geometric (=stable for G) conjugacy classes. In order to lift this to $\tilde{G}$, we need to distinguish elements in the fibers of $\mathbf{p}$.

Definition (J. Adams, 1998)

Let $\tilde{x}, \tilde{y} \in \tilde{G}$ with semisimple regular images $x, y \in G(F)$. They are stably conjugate if

- x, y are stably conjugate,
- $(\mathrm{tr}(\omega_\psi^+) - \mathrm{tr}(\omega_\psi^-))(\tilde{x}) = (\mathrm{tr}(\omega_\psi^+) - \mathrm{tr}(\omega_\psi^-))(\tilde{y})$.

Recall: the characters $\mathrm{tr}(\omega_\psi^\pm)$ are smooth on the set of regular semisimple elements (Harish-Chandra / Maktouf).

Transfer factors

Let $\gamma = (\gamma', \gamma'') \in H(F)$, $\delta \in G(F)$ be semisimple regular such that $\gamma \leftrightarrow \delta$. Take $\tilde{\delta} \in \mathbf{p}^{-1}(\delta)$. Set

$$\Delta(\gamma, \tilde{\delta}) = \Delta'(\tilde{\delta}')\Delta''(\tilde{\delta}'')\Delta_0(\delta', \delta'').$$

Definition de Δ', Δ''. We decompose $W = W' \oplus W''$ according to the eigenvalues coming from γ' and γ'', and $\delta = (\delta', \delta'')$ accordingly (use the regularity of δ).

$$\begin{tikzcd}
(\tilde{\delta}', \tilde{\delta}'') \in & \widetilde{\mathrm{Sp}}(W') \times \widetilde{\mathrm{Sp}}(W'') \arrow[r] \arrow[d, two heads] & \widetilde{\mathrm{Sp}}(W) \arrow[d, two heads] \\
(\delta', \delta'') \in & \mathrm{Sp}(W') \times \mathrm{Sp}(W'') \arrow[r, hook] & \mathrm{Sp}(W)
\end{tikzcd}$$

$(\tilde{\delta}', \tilde{\delta}'')$ is not unique, however the product $\Delta'(\tilde{\delta}')\Delta''(\tilde{\delta}'')$ is. We set

$$\Delta'(\tilde{\delta}') := \frac{(\mathrm{tr}(\omega_\psi^+) - \mathrm{tr}(\omega_\psi^-))(\tilde{\delta}')}{|(\mathrm{tr}(\omega_\psi^+) - \mathrm{tr}(\omega_\psi^-))(\tilde{\delta}')|},$$

$$\Delta'(\tilde{\delta}'') := \frac{(\mathrm{tr}(\omega_\psi^+) + \mathrm{tr}(\omega_\psi^-))(\tilde{\delta}'')}{|(\mathrm{tr}(\omega_\psi^+) + \mathrm{tr}(\omega_\psi^-))(\tilde{\delta}'')|},$$

The Weil representations are taken w.r.t. the spaces $(W', \langle , \rangle)$, $(W'', \langle , \rangle)$ respectively.

Definition of $\Delta_0(\delta', \delta'')$. It is defined using linear algebra and elementary algebraic number theory. Its definition relies on the parametrization of semisimple classes.

- These factors satisfy the properties needed for stabilization: cocycle condition, parabolic descent, product formula, normalization in the unramified case, etc.
- They also satisfy a curious symmetry condition:

$$\Delta_{(n',n'')}((\gamma', \gamma''), \tilde{\delta}) = \Delta_{(n'',n')}((\gamma'', \gamma'), -\tilde{\delta})$$

  where the canonical element $-\tilde{\delta} \in \mathbf{p}^{-1}(-\delta)$ is definable only after pushing-forward by $\mu_2 \hookrightarrow \mu_8$.

Transfer

Fix (n', n'') and the endoscopic group H. Let $J_{\tilde{G}}(\cdots)$ (resp. $J_H^{\mathrm{st}}(\cdots)$) be the orbital integrals on $\tilde{G}$ (resp. stable orbital integrals on H) normalized by Weyl discriminants.

Theorem

$\forall$ antigenuine $f \in C_c^\infty(\tilde{G})$, $\exists f^H \in C_c^\infty(H(F))$ such that

$$\sum_{\delta : \delta \leftrightarrow \gamma} \Delta(\gamma, \tilde{\delta}) J_{\tilde{G}}(\tilde{\delta}, f) = J_H^{st}(\gamma, f^H)$$

for all semisimple G-regular $\gamma \in H(F)$. Here $\tilde{\delta} \in \mathbf{p}^{-1}(\delta)$ is arbitrary.

Remark. If $\delta \leftrightarrow \gamma$, the connected centralizers G_δ, H_γ are isomorphic and we use corresponding Haar measures.

Fundamental lemma for units

Suppose res.char$(F) = p \gg 0$ w.r.t. n and that $\psi : F \to \mathbb{C}^\times$ is of conductor $\mathfrak{o}_F$.
Fix a self-dual lattice $L \subset W$, let $K := \mathrm{Stab}(L)$; it is hyperspecial in $G(F)$ and lifts canonically to $\tilde{G}$. Let f_K be the unit in the corresponding antigenuine spherical Hecke algebra.
Use the Haar measures on $G(F)$ and $H(F)$ such that any hyperspecial subgroup has volume 1.

Theorem

For $f = f_K$, we may take f^H to be the unit in a spherical Hecke algebra of H.

Descent

Reduce to to Lie algebras of the connected centralizers of semisimple elements ($\Rightarrow$ the metaplectic covering disappears!)

1. General machinery: done by Harish-Chandra, Langlands-Shelstad and Waldspurger.
2. Parametrization of semisimple classes in classical groups: identify their centralizers and keep track of the correspondence of conjugacy classes after descent to Lie algebra.
3. Descent of transfer factors: identify the local behavior of Δ', Δ'' and Δ_0, we need
    1. character formulas for ω_ψ (due to Maktouf or Thomas),
    2. elementary manipulations + linear algebra, this is somehow the least trivial part.
4. Apply the endoscopic transfer for symplectic groups, unitary groups and the nonstandard transfer on Lie algebras.

Crucial step: apply the nonstandard transfer

Example: $n' = n$, $n'' = 0$, $H = \mathrm{SO}(2n+1)$

Consider the case $\delta = \exp(X)$, $\gamma = \exp(Y)$ where X, Y are close to 0. Then $\Delta(\exp(Y), \exp(X)) = 1$ for $\exp(Y) \leftrightarrow \exp(X)$, which equivalent to $X \leftrightarrow Y$ (correspondence by nonzero eigenvalues). Thus we are to find $f^{H,\flat} \in C_c^\infty(\mathfrak{h}(F))$ such that

$$J_G^{\mathrm{st}}(X, f^\flat) = J_H^{\mathrm{st}}(Y, f^{H,\flat}).$$

for all $X \in \mathfrak{g}_{\mathrm{reg}}(F)$, $Y \in \mathfrak{h}_{\mathrm{reg}}(F)$ such that $X \leftrightarrow Y$.
This is exactly the nonstandard endoscopic transfer for the triplet $(\mathrm{Sp}(2n), \mathrm{Spin}(2n+1), \ldots)$.

Fundamental lemma: Idem, use topological Jordan decomposition to descend to Lie algebras.

Prospects

1. Stabilization for the elliptic regular part of the trace formula.
2. Generalized fundamental lemma (for all elements in the antigenuine spherical Hecke algebra) $\Rightarrow$ adapt [Hales].
3. Character relations.
4. Other coverings? Cf. [Hiraga-Ikeda] for the case $G = \mathrm{SL}(2)$.

These results are relatively easy – just adapt the existing arguments.
The case $F = \mathbb{R}$ is discussed by Adams and Renard; they also related this to Howe correspondence.

The same notations as in the paragraph on the Fundamental Lemma.

Levi subgroups of the metaplectic group

After pushing-forward $1 \to \mu_2 \to \tilde{G} \to G(F) \to 1$ by $\mu_2 \hookrightarrow \mu_8$, the Levi subgroups (= pull-back by $\mathbf{p}$ of Levi subgroups of $G(F)$) can be canonically written in the form

$$\tilde{M} = \prod_{i \in I} \mathrm{GL}(n_i) \times \widetilde{\mathrm{Sp}}(W^\flat)$$

where $(W^\flat, \langle , \rangle) \subset (W, \langle , \rangle)$ with

$$2 \sum_{i \in I} n_i + \dim W^\flat = \dim W = 2n.$$

The map $\mathbf{p} : \tilde{M} \to M(F)$ is id. on each $\mathrm{GL}(n_i)$, and is the metaplectic covering on $\widetilde{\mathrm{Sp}}(W^\flat)$.

Endoscopic data in general

Roughly speaking, they are elliptic endoscopic data for Levi subgroups of $\tilde{G}$.

$$\begin{array}{ccc} & & \tilde{G} \\ & & \uparrow{\scriptstyle \text{Levi}} \\ M^! & \underset{\text{endo}}{\overset{\text{ell}}{-\!-\!-}} & \tilde{M} \end{array}$$

The elliptic endoscopic data for $\tilde{M}$, their correspondence of classes and transfer factors are defined in the obvious way.

Construction of new endoscopic data

Let $M = \prod_i \mathrm{GL}(n_i) \times \mathrm{Sp}(W^\flat)$.
To an elliptic endoscopic datum of $\tilde{M}$ associated to (m', m''), $2(m' + m'') = 2m = \dim W^\flat$, we take

$$s_0 \in \widehat{\tilde{M}} = \prod_i \mathrm{GL}(n_i, \mathbb{C}) \times \mathrm{Sp}(2m, \mathbb{C})$$

whose components in $\mathrm{GL}(n_i)$ are trivial and the component in $\mathrm{Sp}(2m, \mathbb{C})$ has eigenvalues $+1$ (m' times) and -1 (m'' times).
By imitating Arthur's construction, we may associate an endoscopic datum of $\tilde{G}$ to $s \in s_0 Z^\circ_{\hat{\tilde{M}}} / Z^\circ_{\hat{\tilde{G}}}$, whose endoscopic group is denoted by $G[s]$:

$$\begin{tikzcd} G[s] \arrow[r, dashed, "\text{endo}"] & \tilde{G} \\ M^! \arrow[u, hook, "\text{Levi}"] \arrow[r, dashed, "\text{ell}", "\text{endo}"'] & \tilde{M} \arrow[u, hook, "\text{Levi}"'] \end{tikzcd}$$

Set

$$\mathcal{E}_{M^!}(\tilde{G}) := \{ s \in s_0 Z^{\circ}_{\hat{\tilde{M}}} / Z^0_{\hat{\tilde{G}}} : \text{ the datum is elliptic } \}.$$

This is a finite set.

Caution: the correspondence of classes induced by

and that induced by

are different!

Let K be the hyperspecial subgroup of $G(F)$ associated to a self-dual lattice. Lift it to a subgroup of $\tilde{G}$.

Antigenuine unramified weighted orbital integrals

$$r^{\tilde{G}}_{\tilde{M},K}(\tilde{\delta}) = |D^G(\delta)|^{\frac{1}{2}} \int_{G_\delta(F)\backslash G(F)} f_K(x^{-1}\tilde{\delta}x)v_M(x)\mathrm{d}x.$$

with $\tilde{\delta} \in \tilde{G}$ such that $\delta = \mathbf{p}(\tilde{\delta})$ is semisimple regular. Here v_M is Arthur's weight function.

Here we should fix a Haar measure on $G_\delta(F)$ and a W_0^G-invariant positive-definite form on $\mathfrak{a}_0$.

Endoscopic unramified weighted orbital integral

If $\gamma \in M^!(F)$ is semisimple G-regular (in the obvious sense), set

$$r^{\tilde{G}}_{M^!,K}(\gamma) := \sum_{\substack{\delta \in M(F)/\mathrm{conj} \\ \delta \leftrightarrow \gamma}} \Delta(\gamma, \tilde{\delta}) r^{\tilde{G}}_{\tilde{M},K}(\tilde{\delta}).$$

It will be shown to be independent of the choice of K.

The stable side

For $s \in \mathcal{E}_{M^!}(\tilde{G})$, the following quotient is well-defined

$$i_{M^!}(\tilde{G}, G[s]) := \frac{[Z_{\widehat{M^!}} : Z^{\circ}_{\hat{\tilde{M}}}]}{[Z_{\widehat{G[s]}} : Z^{\circ}_{\hat{\tilde{G}}}]}.$$

Twist

$$\gamma \mapsto \gamma[s] \in M^!(F)$$

This compensates the difference between correspondences of classes.

Weighted fundamental lemma

Recall that Arthur has defined the unramified stable terms $s^{G[s]}_{M^!}(\gamma[s])$.

Theorem

Suppose the nonstandard weighted fundamental lemma on Lie algebra is satisfied for all triplets $(Sp(2m), Spin(2m+1), \ldots)$ *(w.r.t their Levi subgroups) with* $m \leq n$*, then for all semisimple* G*-regular* $\gamma \in M^!(F)$*, we have*

$$r^{\tilde{G}}_{M^!,K}(\gamma) = \sum_{s \in \mathcal{E}_{M^!}(\tilde{G})} i_{M^!}(\tilde{G}, G[s]) s^{G[s]}_{M^!}(\gamma[s]).$$

Method of proof

1. Descend to Lie algebras; this works for both the endoscopic side and the stable side (Waldspurger).
2. Some complicated combinatorial coefficients appear after descent.
3. Apply weighted fundamental lemma and its conjectural nonstandard version on Lie algebras, then express everything in terms of stable terms $s(\cdots)$ on Lie algebras.
4. Compare the combinatorial coefficients (à la Arthur).

Some miracle happens in the last step. This somehow justifies our rule concerning $Z^{\circ}_{\hat{\tilde{G}}}$.