Automorphic Forms and Related Topics

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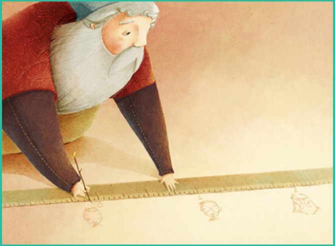

Survey of Basic Functions

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The cover picture of Euclid (nguồn: Internet) is from the book *Ai và Ky ở xứ sở những con số tàng hình* by Nguyễn Phương Văn and Ngô Bảo Châu. Published by Nhã Nam & NXB Thế Giới, 2012.

The ideas of Braverman-Kazhdan

Main references

- R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Springer LNM 260, 1972.
- A. Braverman and D. Kazhdan. *γ-functions of representations and lifting*. In: Geom. Funct. Anal. Special Volume, Part I (2000). With an appendix by V. Vologodsky, GAFA 2000 (Tel Aviv, 1999)
- Ngô Bảo Châu, *On geometry of arc spaces, the Hankel transform and function equation of L-functions*, the 18th Takagi Lectures (2016).
- Sakellaridis' talks, for a broader perspective.

Goal: generalize Godement–Jacquet theory to more general L-functions.

Review of Godement–Jacquet

- F: local field, and $\mathrm{GL}(n) \hookrightarrow M_n$ (more generally: $A^\times \hookrightarrow A$ where A: central simple F-algebra);
- $\mathscr{S}(M_n)$: the space of Schwartz–Bruhat functions on $M_n(F)$;
- for any $\phi \in \mathscr{S}(M_n)$ and $s \in \mathbb{C}$, let $\phi_s := |\det|^s \cdot \phi$;
- for any admissible irreducible representation (π, V) of $\mathrm{GL}(n, F)$, let $\pi_s := |\det|^s \otimes \pi$.

Zeta integral

Let $v \otimes \check{v} \in V_\pi \otimes V_{\check{\pi}}$ and $\xi \in \mathscr{S}(M_n)$.

$$Z^{\mathrm{GJ}}(s, v \otimes \check{v}, \xi) := \int_{\mathrm{GL}(n,F)} \underbrace{\langle \check{v}, \pi(x)v \rangle}_{\text{matrix coefficient}} |\det x|^{s + \frac{n-1}{2}} \xi(x) \, d^\times x.$$

Here $d^\times x$ is a Haar measure and $\Re(s) \gg_\pi 0$.

The case $n = 1$ is Tate's thesis (1950).

1. Meromorphic continuation to all $s \in \mathbb{C}$, rational in q^s for non-archimedean F.
2. Functional equation

$$Z^{\mathrm{GJ}}(1-s, \check{v} \otimes v, \mathscr{F}\xi) = \underbrace{\gamma^{\mathrm{GJ}}(s, \pi)}_{\gamma\text{-factor}} Z^{\mathrm{GJ}}(s, v \otimes \check{v}, \xi);$$

   $\mathscr{F} : \mathscr{S}(M_n) \to \mathscr{S}(M_n)$ is the Fourier transform.
3. In the unramified case with $\xi^\circ := \mathbf{1}_{M_n(\mathfrak{o}_F)}$, $\mathrm{vol}(\mathrm{GL}(n, \mathfrak{o}_F)) = 1$ and $v, \check{v}$ unramified vectors with $\langle \check{v}, v \rangle = 1$, we have

$$Z^{\mathrm{GJ}}(s, v \otimes \check{v}, \xi^\circ) = L(s, \pi).$$

4. In general (for non-archimedean F at least), define $L(s, \pi)$ to be the gcd of zeta integrals, as ξ and $v \otimes \check{v}$ vary.

- Global, adélic counterpart: use

$$\mathscr{S}(M_n) := \bigotimes'_{v\text{:place of } F} \mathscr{S}(M_{n,v}) \quad \text{w.r.t. } \xi_v^\circ$$

  and *Poisson summation* formula on M_n.
- For archimedean F: take Casselman–Wallach representations π. One can also improve the original Godement–Jacquet to obtain continuity properties of zeta integrals in ξ, $v \otimes \check{v}$.
- A more canonical formalism: take $\mathscr{S}$ = Schwartz–Bruhat *half-densities* on M_n. Bonus:

$$Z^{\mathrm{GJ}}(s, \cdots) \rightsquigarrow L\left(\frac{1}{2} + s, \pi\right).$$

Braverman–Kazhdan [2], local case

- G: split connected reductive F-group with Langlands dual $\hat{G}$;
- $\rho : \hat{G} \to \mathrm{GL}(N, \mathbb{C})$ an irreducible representation.

Generalize $(\mathrm{GL}(n) \hookrightarrow M_n, \det)$ into $(G \hookrightarrow X_\rho, \det_\rho)$, where

- X_ρ: normal *reductive monoid* with unit group G, constructed in a canonical fashion from ρ, so that $G \hookrightarrow X_\rho$ is a $G \times G$-equivariant open immersion;
- $\det_\rho : G \twoheadrightarrow \mathbb{G}_m$ comes from the *abelianization map* $X_\rho \twoheadrightarrow \mathbb{G}_a$ of monoid, and is dual to $\mathbb{C}^\times \hookrightarrow \hat{G}$ such that $\rho(z) = z \cdot \mathrm{id}$ for $z \in \mathbb{C}^\times$.

$$Z^{\mathrm{BK}}(s, v \otimes \check{v}, \xi) := \int_{G(F)} \xi(x) \langle \check{v}, \pi(x) v \rangle | \det_\rho(x)|^s \, dx$$

Now ξ is taken from some Schwartz space $\mathscr{S}_\rho$, and $\Re(s) \gg_\pi 0$, with π, $v \otimes \check{v}$ are as before.

We seek to construct $\mathscr{S}_\rho$, and a basic function $\xi^\circ \in \mathscr{S}_\rho$ in the unramified case.

Desiderata. Cf. Ngô's Takagi talk

- Meromorphic/rational continuation of $Z^{\mathrm{BK}}(s, \ldots)$.
- Functional equation via some *Hankel transform* $\mathscr{S}_\rho \to \mathscr{S}_\rho$ preserving ξ°.
- As in the Godement–Jacquet case, we require that

$$Z^{\mathrm{BK}}(s, v \otimes \check{v}, \xi^\circ) = L(s + ?, \pi, \rho), \quad \langle \check{v}, v \rangle = 1.$$

  in the unramified case, with some explicit shift in s......

The basic function ξ° is closely related to the singularities of the *monoid* X_ρ.

Inverse Satake transform

Let G be a split group over F: non-archimedean local field, $q := \#\mathfrak{o}_F/\mathfrak{p}_F$. Let $K := G(\mathfrak{o}_F)$. The *spherical Hecke algebra* (precisely: of measures) is

$$\mathscr{H} := C_c^\infty(K\backslash G(F)/K; \mathbb{C}) + \star, \quad \mathrm{vol}(K) = 1.$$

Classical Satake isomorphism

Take T: Cartan torus, W: Weyl group. The "constant term" map yields

$$\mathrm{Sat} : \mathscr{H} \xrightarrow{\sim} \mathbb{C}[\hat{T}]^W = \mathbb{C}[\hat{T} /\!/ W].$$

The last term is also $\mathbb{C}[\hat{G} /\!/ \hat{G}]$ or $\mathrm{Rep}(\hat{G}) \otimes \mathbb{C}$.

One can replace $\mathbb{C}$ by $\mathbb{Z}[q^{\pm 1/2}]$, or work in the ℓ-adic setting.

Basic functions for monoids

- The K-unramified irreducible representations π of $G(F)$ are in bijection with elements of $\hat{T} /\!\!/ W$.
- For every s, the $\pi \mapsto L(s, \pi, \rho)$ defines an element in the formal completion of $\mathbb{C}[\hat{T} /\!\!/ W]$ relative to some explicit cone.
- The property $Z^{\mathrm{BK}}(s, v \otimes \check{v}, \xi^\circ) = L(s, \pi, \rho)$ turns out to be equivalent to
$$\xi^\circ = \sum_{n \geq 0} \xi_n, \quad \mathrm{Supp}(\psi_n) \subset_{\mathrm{cpt}} G(F)^{v \circ \det_\rho = n},$$
such that
$$\xi_n \xrightarrow{\mathrm{Sat}} \mathrm{Tr}\left(\mathrm{Sym}^n \rho\right) \in \mathrm{Rep}(\hat{\mathrm{G}}).$$

Not so easy to describe ξ° by inverting Sat explicitly: some Kazhdan–Lusztig polynomials for affine Weyl groups will appear.

Approach 1 for basic functions

The structure of Sat^{-1} being combinatorial, one can assume char$(F) = p > 0$ and work ℓ-adically, $\ell \neq p$.
Let $\mathscr{L}X$ be the *formal arc space* of (X: scheme of finite type over $\mathbb{F}_q$).
It is the functor

$$\mathscr{L}X(R) = X(R[\![t]\!]), \quad \forall \mathbb{F}_q - \text{algebra } R.$$

Thus $\mathscr{L}X(\mathbb{F}_q) = X(\mathbb{F}_q[\![t]\!]) = X(\mathfrak{o}_F)$.
Function-sheaf dictionary. Interesting constructible sheaves on $Z_{/\mathbb{F}_q}$ (finite type) $\rightsquigarrow$ interesting functions on $Z(\mathbb{F}_q)$. Eg. the IC-complex $\mathrm{IC}_Z \rightsquigarrow$ IC-function. Normalize IC_Z to be $\mathbb{Q}_\ell$ on the smooth stratum.[1]

Philosophy

$\mathrm{IC}_{\mathscr{L}X_\rho} \rightsquigarrow \xi^\circ$. Obstacle: $\mathscr{L}X_\rho$ is REALLY infinite-dimensional.

[1] Sakellaridis remarked that it is not the most reasonable choice.

- When X_ρ is smooth (eg. Godement–Jacquet), it is reasonable to expect that $\mathrm{IC}_{\mathscr{L}X_\rho} = \mathbb{Q}_\ell$; the IC-function is then $\mathbf{1}_{X_\rho(\mathfrak{o}_F)}$.
- Grinberg–Kazhdan–Drinfeld: finite-dimensional models of the singularities of $\mathscr{L}Z$ (for $Z_{/\mathbb{F}_q}$ of finite type). This leads to a general definition of IC-functions and even IC-sheaves on $\mathscr{L}Z$.
- By [1], $\xi^\circ := \mathrm{IC}_{\mathscr{L}X_\rho}$ points to $L(s - \langle \eta_G, \lambda \rangle, \pi, \rho)$ instead of $L(s, \pi, \rho)$. Here η_G is the half-sum of positive roots, and λ is the highest weight of ρ.
  The local-global argument will be reviewed later.

A. Bouthier, B. C. Ngo, Y. Sakellaridis. *On the formal arc space of a reductive monoid*. Amer. J. Math. 138 (2016), no. 1, 81–108. With erratum.

A. Bouthier and D. Kazhdan. *Faisceaux pervers sur les espaces d'arcs I: le cas d'égales caractéristiques*. Preprint, 2015,

Approach 2 for basic functions

Let (V, ρ) be the representation of $\hat{G}$ in question. Write

$$\xi^\circ_Y = \sum_\mu c_\mu(q) \delta_{B^-}(\mu(\varpi))^{\frac{1}{2}} \mathbf{1}_{K\mu(\varpi)K} Y^{\det_\rho(\mu)}$$

where Y is a variable, μ ranges over the anti-dominant part of $X_*(T)$, and $\det_\rho : X_*(T) \to \mathbb{Z}$.
Substitution $Y \rightsquigarrow q^{-s}$ yields $\xi^\circ |\det_\rho|^s$. We want to describe c_μ.

- $\hat{\mathfrak{b}} = \hat{\mathfrak{t}} \oplus \hat{\mathfrak{u}}$ is the dual Borel with adjoint action of $\hat{B}$.
- Ψ: the multi-set of $\hat{T}$-weights on $V \oplus \hat{\mathfrak{u}}$ (as a $\hat{B}$-representation).
- ρ^-: the half-sum of negative coroots.

$$m^\mu_{\lambda,\Psi}(q) := \sum_{w \in W} (-1)^{\ell(w)} \mathscr{P}_\Psi \left(w(\lambda + \rho^-) - (\mu + \rho^-); q \right)$$

Here $\mathscr{P}_\Psi$ is the q-analogue of Kostant's partition function

$$\prod_{\alpha \in \Psi} (1 - qe^{-\alpha})^{-1} = \sum_{\nu \in X_*(T)} \mathscr{P}_\Psi(\nu; q) e^\nu.$$

Theorem

For all anti-dominant μ we have

$$c_\mu(q^{-1}) = q^{-\det_\rho(\mu)} m^\mu_{0,\Psi}(q).$$

This is done by elementary invariant-theoretic arguments on $\hat{G}$, based on prior works of Broer, R. Brylinski, ...

Note: $m^{\mu}_{\lambda,\Psi}$ is a special case of *generalized Kostka–Foulkes polynomials* (Panyushev).

Many of its properties can be deduced from the geometry of the $\hat{G}$-equivariant vector bundle $\hat{G} \times^{\hat{B}} (V \oplus \hat{\mathfrak{u}}) \twoheadrightarrow \hat{G}/\hat{B}$.

L. *Basic functions and unramified local L-factors for split groups*, Science China: Mathematics, Vol 60, No.5, 2017, pp.777-812.

Nevertheless...

The formula in the Theorem is not convenient for computation when $\mathrm{rank}(G)$ is large.

Approach 3 for basic functions (sketch)

In [1], Sakellaridis gave another recipe for inverting the Satake transform, in the more general setting that

- Z: affine spherical homogeneous G-space,
- moreover, assume $\hat{A}_Z = \hat{A}_{Z,\mathrm{Gaitsgory-Nadler}}$.

$$\mathrm{Sat}^{-1} : \mathbb{C}\left[\delta_{(Z)}^{\frac{1}{2}} \hat{A}_{Z,\mathrm{G-N}}\right]^{W_Z} \longrightarrow C_c^\infty(Z(F))^K$$

Main techniques: boundary degeneration + theory of spherical functions on Z.
The group case $Z = G$ with $G \times G$-action: one can use this to "invert L-factors", and recovers the formula in terms of $m^{\mu}_{0,\Psi}(q)$.

Y. Sakellaridis. *Inverse Satake transforms* arXiv:1410.2312

Digression: The works of Finkelberg--Ionov and Hu

Known (Achar–Hendeson, Finkelberg–Ginzburg–Travkon): The IC-stalks of $\mathscr{L}\mathrm{GL}(N)$-orbit closures in the affine mirabolic Grassmannian yields the *Kostka–Shoji polynomials*.
Multi-variable case: Finkelberg–Ionov arXiv:1605.05806

$$K_{\vec{\lambda},\vec{\mu}}(t_1, \ldots, t_r) := \sum_{\sigma \in \mathfrak{S}_N^r} (-1)^{\sigma} L_r^{\sigma(\vec{\lambda}+\vec{\rho})-(\vec{\mu}+\vec{\rho})}(t_1, \ldots, t_r)$$

- $\vec{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$, with $\lambda^{(s)} = [\lambda_1^{(s)} \geq \cdots \geq \lambda_N^{(s)}]$ (integers), same for $\vec{\mu}$;
- $\vec{\rho} = (\rho, \ldots, \rho)$ where $\rho = (N, \ldots, 2, 1)$;
- $L_r^{\vec{\alpha}}(t_1, \ldots, t_r)$: partition function of the *pseudo-roots* $\alpha_{mn} = \sum_{l=m}^{n-1} \delta_l$ where $1 \leq m < n \leq rN$, $n - m \equiv 1 \pmod{r}$, and $\{\delta_l\}_l$ is the natural basis of $\mathbb{Z}^{rN-1}$; each occurrence of α_{mn} contributes a t_s if $m \equiv s \pmod{r}$, where $s \in \{1, \ldots, r\}$.

- When $r = 2$ and $t_1 = t_2 =: t$, we recover the Kostka–Shoji polynomials.
- Expectation: $K_{\vec{\lambda},\vec{\mu}}(t_1, \dots, t_r) \in \mathbb{Z}_{\geq 0}[t_1, \dots, t_d]$. This will follow from a vanishing property of $H^{>0}$ of certain line bundles $\mathscr{O}(\vec{\mu})$ on

$$\mathscr{T}_r{}^*\mathscr{B}_N^r := \mathrm{GL}_N^r \times^{B_N^r} \mathfrak{n}_r$$

  for some B_N^r-representation $\mathfrak{n}_r$. When $r = 1$ we get $\mathscr{T}^*\mathscr{B}_N$.

Here $B_N \subset \mathrm{GL}_N$ is the Borel subgroup; both are taken over $\mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$. This method is due to Brylinski and Broer.

- Recently, the required vanishing is proven by Yue Hu arXiv:1704.07947. In characteristic zero, it can be deduced from Panyushev's work.
- The scenario is also similar to that in the study of *basic functions* in the Approach 2.
- It would require some innovations to relate $K_{\vec{\lambda},\vec{\mu}}(t_1, \dots, t_r)$ to IC-stalks.

More general local zeta integrals

Generalize Braverman–Kazhdan by integrating a product of

1. suitable Schwartz functions on an affine normal X with a Zariski–open G-orbit X^+, such that $X \smallsetminus X^+ = \{f = 0\}$ for some G-eigenfunction f;
2. coefficients of an admissible representation π of $G(F)$ on $X^+(F)$, in the sense of *relative harmonic analysis*.

Assume X^+ is spherical and wavefront. Such integrals can be twisted by $|f|^s$ and we *expect* meromorphic continuation in s. Local functional equations are *hopefully* reflected by equivariant isomorphisms between Schwartz spaces.

L. *Zeta integrals, Schwartz spaces and local functional equations*. arXiv:1508.05594.

Needed: accessible examples!

Case study: prehomogeneous vector spaces

1. As in Braverman–Kazhdan program, singularity creates difficulties. The easiest case: X is a vector space, on which G acts linearly — the definition of Schwartz functions is then standard (for monoids, the only possibility is Godement–Jacquet).
2. Then X is a *prehomogeneous vector space*. Zeta integrals of this type with $\pi = \mathbf{1}$ have a long history (Shintani, Igusa...) and are closely related to geometry. Eg. the poles are related to the log-canonical threshold of f at $\vec{0}$.

Several results in the prehomogeneous case have been obtained for non-archimedean F.
When $F = \mathbb{R}$ and X^+ is essentially a symmetric G-space $\rightsquigarrow$ the meromorphic continuation of the zeta integrals "with coefficients".

L. *Towards generalized prehomogeneous zeta integrals*. arXiv:1610.05973.

Case study: the doubling method

Let G: a classical group, and π: irreducible admissible representation of $G(F)$. Consider

- matrix coefficients c of π, against
- "good sections" f of a degenerate parabolic induction $I_P^{G^\square}(\chi)$, where $G^\square$ is a *bigger group* and P is a parabolic, and that

$$G \times G \hookrightarrow G^\square,$$

$$G = G \times \{1\} \to P\backslash G^\square : \ G \times G\text{-equivariant, open embedding}$$

Here $P \twoheadrightarrow M$ is the Levi quotient, and χ is a character of $M_{\mathrm{ab}}(F)$. [2]

Piatetski–Shapiro-Rallis [1]

The integration of c against f (pulled back to $G = G \times \{1\}$) over $G(F)$ yields the standard local L-factor for $\chi \times \pi$, upon some "sufficiently positive" twist on χ.

[2] We omit the other requirements on these data.

- What is more important is the quasi-affine $G^{\square}$-space $X_P := P_{\mathrm{der}} \backslash G^{\square}$.
- We have a natural right $M_{\mathrm{ab}} \times G^{\square}$-action on X_P

$$P_{\mathrm{der}} x \xmapsto{(m,g)} P_{\mathrm{der}} m^{-1} x g.$$

- The doubling method is rephrased in [2] using *Schwartz functions* on the affine closure X of X_P, as *universal good sections*. See also [Getz–Liu].

[1] I. Piatetski-Shapiro and S. Rallis. *ϵ-factor of representations of classical groups*. Proc. Nat. Acad. Sci. U.S.A., 83(13), 1986.

[2] A. Braverman and D. Kazhdan. *Normalized intertwining operators and nilpotent elements in the Langlands dual group*. In: Mosc. Math. J. 2.3 (2002).

To simplify matters, take $G = \mathrm{Sp}(W)$, $G^{\square} = \mathrm{Sp}(W \oplus \overline{W})$ where $\overline{W}$ is W with $-\langle , \rangle$. Let $P \subset G^{\square}$ be the stabilizer of the Lagrangian $\mathrm{diag}(W)$.

Doubling vs. monoid

Let X^+ be the open $M_{\mathrm{ab}} \times G \times G$-orbit in X. There is a natural $M_{\mathrm{ab}} \times G \times G$-equivariant embedding

$$M_{\mathrm{ab}} \times G \simeq X^+ \subset X := \overline{X_P}^{\mathrm{aff}}.$$

Fact (Rittatore). Such an equivariant normal affine embedding automatically makes X into a *monoid* with unit group $M_{\mathrm{ab}} \times G$. It is *flat* in Vinberg's sense, whose *abelianzation map* restricts to

$$c : M_{\mathrm{ab}} \times G \twoheadrightarrow M_{\mathrm{ab}} = \mathrm{GL}(W)_{\mathrm{ab}} \xrightarrow{\det^{-1}} \mathbb{G}_m.$$

Identify M_{ab} with $\mathbb{G}_m$ by $\det^{-1}$.

Relation to the L-monoids of Ngô *et al.*

The monoid X is the L-monoid associated to the $(2n+1)$-dimensional representation $\mathrm{id} \boxtimes \mathrm{std}$ of $(\mathbb{G}_m \times G)^\wedge$.

1. We obtain another accessible case of Braverman–Kazhdan program of L-monoids: the Schwartz spaces and Fourier/Hankel transforms are already available.
2. Shahidi and Getz–Liu (2017) pursued these ideas further.
3. In particular, Braverman and Kazhdan defined a *basic function* $c_P = c_{P,0}$ from the geometry of $X_P \subset X$.
   Combinatorial formulas for c_P exist, but the ultimate motivation seems to come from $\mathrm{IC}_{\overline{\mathrm{Bun}_P}}$ (here: pass to equal-characteristics).

Idea: Over $\mathbb{F}_q$, *Drinfeld's compactification* $\overline{\mathrm{Bun}_P} \supset \mathrm{Bun}_P$ serves as a global model of the singularities of $\mathscr{L}X$.

Let C be a complete, smooth, geometrically connected curve over $\mathbb{F}_q$, and $[\cdots]$ = the quotient stacks; $X \supset X_P \supset X^+$ are all defined over $\mathbb{F}_q$.

$$\overline{\mathrm{Bun}_P} = \left\{ C \xrightarrow{\phi} \left[\frac{X}{M_{\mathrm{ab}} \times G^{\square}} \right] : \text{image } \subset \left[\frac{X_P}{M_{\mathrm{ab}} \times G^{\square}} \right] \text{ generically} \right\},$$

as an open substack of $\mathrm{Map}\left(C, \left[\frac{X}{M_{\mathrm{ab}} \times G^{\square}} \right] \right)$ over $\mathbb{F}_q$.

- More precisely, we should prescribe a groupoid $\overline{\mathrm{Bun}_P}(S)$ for test schemes $S_{/\mathbb{F}_q}$.
- As explained in [Bouthier–Kazhdan] [3], one can relate c_P or $\mathrm{IC}_{\overline{\mathrm{Bun}_P}}$ to $\mathrm{IC}_{\mathscr{L}X}$.

[3] We only need the last section thereof; the idea is similar to the case of monoids in [Bouthier–Ngô–Sakellaridis], cf. the next slide.

1. Viewing X as a monoid of unit group $X^+ \simeq G' := M_{\mathrm{ab}} \times G$, there is another *basic function* $\mathbb{L}$ associated to X.
2. Bouthier–Ngô–Sakellaridis: $\mathbb{L}$ as the IC-function of $\mathscr{L}X$, up to a "shift" $s \rightsquigarrow s + n$. This is obtained by relating $\mathrm{IC}_{\mathscr{L}X}$ to $\mathrm{IC}_{\mathscr{M}}$ of the global model

$$\mathscr{M} = \left\{ C \xrightarrow{\phi} \left[\frac{X}{G' \times G'} \right] : \text{image } \subset \left[\frac{X^+}{G' \times G'} \right] \text{ generically} \right\}.$$

By relating both c_P and $\mathbb{L}$ to the IC-function of $\mathscr{L}X$ in the equal-characteristic case, we obtain

A comparison of basic functions, appendix to Shahidi's preprint

As functions on $X^+ \simeq M_{\mathrm{ab}} \times G$, we have $\mathbb{L} = c_P |c|^n$, where c is the abelianization map $X^+ \to \mathbb{G}_m$.

The shift in $\mathbb{L} = c_P |c|^n$ explained

Key: The Schwartz spaces proposed by Braverman–Kazhdan are always $\subset L^2(X)$. We are thus led to consider Schwartz *half-densities* (= square-roots of measures): what is basic about $X_P \subset X$ (resp. $X^+ \subset X$) is $c_P |\Xi|^{\frac{1}{2}}$ (resp. $\mathbb{L}|\Omega|^{\frac{1}{2}}$), where

1. $|\Xi|$ is a $G^\square$-invariant measure on $X_P(F) = P_{\mathrm{der}}(F)\backslash G^\square(F)$ (note: $|\Xi|$ is not $M_{\mathrm{ab}}(F)$-invariant);
2. $|\Omega|$ is a $M_{\mathrm{ab}} \times G \times G$-invariant measure on $(M_{\mathrm{ab}} \times G)(F) \simeq X^+(F)$, suitably normalized.

We may restrict half-densities from $X_P(F)$ to its open subset $X^+(F)$.

If you prefer central L-values, $\mathbb{L}\left(\frac{1}{2}\right) := \mathbb{L}|c|^{\frac{1}{2}}$ is even more basic than $\mathbb{L}$.

Fact: one can actually take $|\Omega| = |c|^{-2n-1}|\Xi|$.

Hence $\mathbb{L} = c_P|c|^n$ is equivalent to the equality of half-densities

$$c_P|\Xi|^{\frac{1}{2}} = c_P|c|^{n+\frac{1}{2}}|\Omega|^{\frac{1}{2}} = \mathbb{L}\left(\frac{1}{2}\right)|\Omega|^{\frac{1}{2}}.$$

In other words, basic = basic.

For details, see the appendix to arXiv:1710.06841.