Errata for *Zeta integrals, Schwartz spaces and local functional equations*

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The numbering below follows the published version in the Lecture Notes in Mathematics, No. 2228 published in 2018 by Springer (ISBN: 978-3-030-01287-8). These corrections are incorporated into the arXiv version ≥ 5 (arXiv:1508.05594).

Page 47, line -8 In the description of the topology on $C_\Omega(X^+)$ appeared an undefined norm $\|\cdot\|$ on the fibers of $\mathcal{E}$ over Ω. It suffices to take any continuous family over Ω of such norms, and the choice is irrelevant for the topology, as Ω is compact.

Page 99, (7.4) $G \hookrightarrow X_P^\flat$.

Page 108, Lemma 7.4.5 The proof of the first assertion is incorrect, and the second assertion is unnecessary. This Lemma is only used in the proof of Theorem 7.4.7 (page 111). Specifically, I used it to argue that for any fundamental weight ϖ of G with respect to a Borel pair (B, T), the defining equation $f_\varpi \in F[X^+]$ of the color D_ϖ in X^+ satisfies

$$v_{\partial X}(tf_\varpi) = 0,$$

so that $tf_\varpi \in F[X]$ (recall that X is normal and ∂X is a prime divisor) and it cuts out $\overline{D_\varpi} \subset X$. Below is a corrected argument for it.

Since $t \in F[X^+] \subset F(X)$ is a uniformizer for ∂X (Lemma 7.2.5, 7.2.6), our goal is to prove $v_{\partial X}(f_\varpi) = -1$ for each fundamental weight ϖ.

Write $\mathbb{G}_m = \operatorname{Spec} F[s, s^{-1}]$. By the proof of Lemma 7.2.6, for $g \in M_{\mathrm{ab}} \times G \times G$ in general position we have

$$\begin{aligned} v_{\partial X}(f_\varpi) &= \operatorname{ord}_{s=0} \left(f_\varpi(c(s)g) \right) \cdot i(c(0), \partial X \cdot c; X) \\ &= \operatorname{ord}_{s=0} \left(f_\varpi(x_0(1, \mu(s))g) \right) \cdot i(c(0), \partial X \cdot c; X), \end{aligned}$$

where the base point $x_0 \in X^+(F)$, the morphism $c : \mathbb{G}_m \to X^+$ and $\mu : \mathbb{G}_m \to G$ are defined in the cited proof, namely $c(s) = sx_0(1, \mu(s))$. Here we used the fact that f_ϖ is invariant under M_{ab}-action. For the same reason, it suffices to take $g = (1, g_1, g_2)$ with generic $g_1, g_2 \in G$.

In the proof of Lemma 7.2.6, the local intersection number $i(c(0), \partial X \cdot c; X)$ has been shown to be 1.

By Lemma 7.2.3, we identify X^+ with $\mathbb{G}_m \times G$ so that $x_0 = (1,1)$; the $\mathbb{G}_m \times G \times G$-action is also specified there. Let ρ be the (right) G-representation of highest weight ϖ; take vectors v_ϖ and $\check{v}_{-\varpi}$ in ρ and $\check{\rho}$, with weights ϖ and $-\varpi$ respectively such that $\langle \check{v}_{-\varpi}, v_\varpi \rangle = 1$. Then

$$f_\varpi(g) = \langle \check{v}_{-\varpi}, \rho(g) v_\varpi \rangle, \quad g \in G.$$

Thus we have to calculate, for generic $g_1, g_2 \in G$:

$$\mathrm{ord}_{s=0} \left\langle \check{\rho}(g_2)\check{v}_{-\varpi}, \rho(\mu(s)^{-1})\rho(g_1) v_\varpi \right\rangle.$$

Consider the standard representation of G on $V = \bigoplus_{i=1}^{n} (Fe_k \oplus Fe_{-k})$. Recall that [1] the fundamental weights of G are $\varpi_k = \epsilon_1^* + \cdots + \epsilon_k^*$ with $k = 1, \ldots, n$; the corresponding highest weight representations ρ_k are realized on

$$E_k := \langle e_{a_1} \wedge \cdots e_{a_l} \wedge e_{-b_1} \wedge \cdots \wedge e_{-b_m} : 1 \leq a_1 < \cdots < b_m \leq n, \ l + m = k \rangle \\ \subset \wedge^k V, \quad k = 1, \ldots, n.$$

We then take

$$v_{\varpi_k} = e_1 \wedge \cdots \wedge e_k, \quad \check{v}_{-\varpi_k} = e_1^* \wedge \cdots \wedge e_k^* \mod E_k^\perp.$$

In the proof of Lemma 7.2.6 we took

$$\mu(s) : e_{\pm i} \mapsto \begin{cases} s^{\pm 1} e_{\pm i}, & i = 1, \\ e_{\pm i}, & i > 1. \end{cases} \quad (1 \leq i \leq n)$$

Using this realization, it is clear that

$$\delta(s, g_1, g_2) := \left\langle \check{\rho}_k(g_2)\check{v}_{-\varpi_k}, \rho_k(\mu(s)^{-1})\rho_k(g_1) v_{\varpi_k} \right\rangle \in s^{-1}F[s],$$

and $\mathrm{ord}_{s=0}(\delta(s, g_1, g_2))$ for generic (g_1, g_2) equals actually

$$\inf_{(g_1, g_2) \in G^2} \mathrm{ord}_{s=0}(\delta(s, g_1, g_2)), \quad \text{which is} \geq -1.$$

Taking $g_1 = 1 = g_2$, the right hand side does attain -1. This completes the proof of $v_{\partial X}(f_\varpi) = -1$.

Alternatively, if the right hand side is ≥ 0 then $f_\varpi \in F[X]$ and is invariant under the $\mathbb{G}_m$-dilation on X. This would imply the constancy of f_ϖ by Lemma 7.4.5. Contradiction.

[1] See N. Bourbaki, *Lie groups and Lie algebras, Chapters 7—9*, pp.206–207.