Notes on the finiteness theorem of Faltings for abelian varieties

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November 14, 2018

Abstract

These are informal notes prepared for the seminar on Faltings' proof of the Mordell conjecture organized by Xinyi Yuan and Ruochuan Liu at Beijing International Center for Mathematical Research, Fall 2018. We follow Faltings–Chai to sketch a proof of the following fact: given g, d, B, there are at most finitely many principally polarized abelian varieties A (up to isomorphisms) over a number field F with $[F:\mathbb{Q}] \leq d$, $\dim A = g$ and whose modular (or Faltings) height satisfies $h_{\mathrm{mod}}(A) \leq B$.

The schemes under considerations are assumed to be separated of finite type over their bases. If $X \to S$ and $R \xrightarrow{f} S$ are morphisms of schemes, we write X_R or $X_{R,f}$ for $X \underset{S,f}{\times} R$, and $X_A := X_{\operatorname{Spec} A}$ if $R = \operatorname{Spec} A$ for some commutative ring A. The same convention pertains to vector bundles, coherent sheaves, etc.

For a local or number field K, we write $\mathfrak{o}_K$ for its ring of integers. As usual, a place of a global field means an equivalence class of valuations. The completion of K at a place v is denoted by K_v. The notation $w \mid v$ stands for extension of places under finite extensions of fields. Therefore, $v \mid \infty$ means that v is an Archimedean valuation.

Our goal: The Main Theorem 5.6.

References: Apart from the notes of the Michigan Seminars on the Mordell Conjecture, I also benefited from [Del85], [FC90] and [Fal+92]. I am deeply grateful to the comments of Xinyi Yuan, Ruochuan Liu and the other participants; all errors are my own responsibility.

1 Metrized line bundles

Let K be a number field. Denote by $|\cdot|_v : K_v \to \mathbb{R}_{\geq 0}$ the corresponding normalized absolute value. Note that for $v \mid \infty$, we take

$$|z|_v = \begin{cases} \text{the usual } |z|, & K_v \simeq \mathbb{R} \\ z\bar{z}, & K_v \simeq \mathbb{C} \end{cases}$$

so that the product formula $\prod_v |x|_v = 1$ holds for all $x \in K^\times$.

Definition 1.1. Let v be a place of K. Let V be a K_v-variety and $\mathcal{L}$ a line bundle over V. A v-adic metric on $\mathcal{L}$ is a family of functions $\|\cdot\|_{v,x} : \mathcal{L}_x \to \mathbb{R}_{\geq 0}$ on the fibers of $\mathcal{L}$, where x ranges over $V(K_v)$, satisfying

$$\ell = 0 \iff \|\ell\|_{v,x} = 0,$$
$$\|t\ell\|_{v,x} = |t|_v \cdot \|\ell\|_{v,x}, \quad t \in K_v, \ \ell \in \mathcal{L}_x.$$

They are assumed to vary continuously in x. When $v \nmid \infty$, we impose the ultrametric inequality on each $\|\cdot\|_{v,x}$. When $v \mid \infty$, we require that there exists an Hermitian form $(\cdot|\cdot)_{v,x}$ (= quadratic form if $K_v = \mathbb{R}$) on $\mathcal{L}_x$ satisfying

$$\|\ell\|_{v,x} = \begin{cases} (\ell|\ell)_{v,x}^{1/2}, & K_v \simeq \mathbb{R} \\ (\ell|\ell)_{v,x}, & K_v \simeq \mathbb{C}; \end{cases}$$

Notice that the Archimedean places of K are in bijection with $\mathrm{Hom}_{\mathrm{ring}}(K, \mathbb{C})$ modulo complex conjugation. Therefore the Archimedean datum above is the same as a family of Hermitian forms $\|\cdot\|_{\iota,x}$ on the fibers of $\mathcal{L}_{\mathbb{C},\iota}$ over $X_{\mathbb{C},\iota}$, indexed by $\iota : K \hookrightarrow \mathbb{C}$, such that

$$(\ell_1|\ell_2)_{\iota,x} = (\overline{\ell_1}|\overline{\ell_2})_{\bar{\iota},\bar{x}}$$

for all $\iota : K \hookrightarrow \mathbb{C}$ with $\iota(K) \subsetneq \mathbb{R}$ and $x \in X_{\mathbb{C},\iota}$.

Definition 1.2. Let X be a K-variety. A *metrized line bundle* on X consists of a line bundle $\mathcal{L}$, a family $\|\cdot\| = (\|\cdot\|_v)_v$ of v-adic metrics on $\mathcal{L}_{K_v}$ for each place v of K.

There is an obvious notion of isomorphisms between metrized line bundles over X. If $(\mathcal{L}_i, \|\cdot\|_i)$ over X are given for $i = 1, 2$, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ equipped $\|\cdot\|_1 \otimes \|\cdot\|_2$ is also metrized. Same for the dual line bundles $\mathcal{L}^\vee$. These operations pass to isomorphism classes.

In practice, the v-adic metrics for $v \nmid \infty$ often come from a $\mathfrak{o}_K$-model of $\mathcal{L}$ and X. Indeed, an integral model gives rise to a "lattice" $\mathcal{L}^\circ \subset \mathcal{L}$. For each $v \nmid \infty$ and $\ell \in \mathcal{L}_x$, we put

$$\|\ell\|_v := \inf \left\{ |t|_v : t \in K_v \text{ s.t. } \ell \in t\mathcal{L}_x^\circ \right\}.$$

So it remains to specify the Archimedean data. This recipe is compatible with taking tensor products and duals. Summing up, a metrized line bundle $\widehat{\mathcal{L}} := (\mathcal{L}, \|\cdot\|)$ can be deduced from

- a *line bundle* $\mathcal{L}$ over X,
- an *integral structure* on $\mathcal{L}$ and X, i.e. $\mathcal{L}$ is actually a line bundle over an $\mathfrak{o}_K$-model $\mathcal{X}$ of X,
- a family of *Hermitian metrics* $\|\cdot\|_v$ for each $v \mid \infty$, varying continuously over $X(K_v)$ and compatible with complex conjugation.

Our future applications require only the following simple case, with $X = \mathrm{Spec}(K)$, $\mathcal{X} = \mathrm{Spec}\, \mathfrak{o}_K$.

Definition 1.3. Let $\mathcal{L}$ be an invertible $\mathfrak{o}_K$-module, and for each embedding $\iota : K \to \mathbb{C}$, we are given an Hermitian form $(\cdot|\cdot)_\iota$ on $\mathcal{L}_{\iota,\mathbb{C}}$ satisfying

$$(\ell_1|\ell_2)_\iota = (\overline{\ell_1}|\overline{\ell_2})_{\bar{\iota}}$$

for all ℓ_1, ℓ_2 in $\mathcal{L}_{\iota,\mathbb{C}}$. Call these data a *metrized line bundle over* $\mathrm{Spec}\, \mathfrak{o}_K$.

Using the Hermitian metrics together with the integral structure, these data give rise to a metrized line bundle $\widehat{\mathcal{L}} := (\mathcal{L}, \|\cdot\|)$ over $\mathrm{Spec}\, K$.

Definition 1.4 (Arakelov degree)**.** For a metrized line bundle $\widehat{\mathcal{L}}$ over Spec $\mathfrak{o}_K$, we put

$$\deg \widehat{\mathcal{L}} := -\log \prod_{v:\text{places}} \|\ell\|_v$$

where $\ell \in \mathcal{L}_K \setminus \{0\}$ is arbitrary (by the product formula). The infinite product makes sense because $\|\ell\|_v = 1$ for almost all $v \nmid \infty$ when ℓ is given. If we take $\ell \in \mathcal{L} \setminus \{0\}$, then

$$\deg \widehat{\mathcal{L}} = -\log \prod_{v|\infty} \|\ell\|_v + \log \left| \frac{\mathcal{L}}{\mathfrak{o}_K \cdot \ell} \right|.$$

Remark 1.5. As a motivation, observe that in the function field case $K = \mathbb{F}_q(C)$, where C is a complete smooth $\mathbb{F}_q$-curve, and a line bundle $\mathcal{L}$ over C, the same recipe yields the usual degree $\deg \mathcal{L} = \dim_{\mathbb{F}_q} (O_C / O_C \cdot f)$ whenever $f \in \Gamma(C, \mathcal{L})$.

Proposition 1.6. *Given any metrized line bundle $\widehat{\mathcal{L}}$ over $\mathfrak{o}_F$ and a finite extension $K|F$ of fields, $\mathcal{L} \underset{\mathfrak{o}_F}{\otimes} \mathfrak{o}_K$ is also metrized in a natural way, and*

$$\deg \widehat{\mathcal{L} \underset{\mathfrak{o}_F}{\otimes} \mathfrak{o}_K} = [K : F] \deg \widehat{\mathcal{L}}.$$

Proof. Recall the identity $[K : F] = \sum_{w|v} [K_w : F_v]$, where v is any place of F. □

2 Arakelov heights

Henceforth K is a number field.

Let X be a projective K-variety with a proper $\mathfrak{o}_K$-model $\mathcal{X}$. Consider a line bundle $\mathcal{L}$ + Hermitian metrics for $v \mid \infty$ + integral structure relative to $\mathcal{X}$, denoted as $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_v, \ldots)$. Every point $x \in X(K)$ extends uniquely to a morphism Spec $\mathfrak{o}_K \to \mathcal{X}$ by the valuative criterion. Therefore $x^* \mathcal{L}$ becomes a metrized line bundle over Spec $\mathfrak{o}_K$.

Definition 2.1 (Arakelov height)**.** In the scenario above, put

$$h_{\widehat{\mathcal{L}}}(x) := \frac{1}{[K : \mathbb{Q}]} \deg x^* \mathcal{L}.$$

The factor $[K : \mathbb{Q}]^{-1}$ ensures that $h_{\widehat{\mathcal{L}}}(x) = h_{\widehat{\mathcal{L}}_E}(x)$ for any finite extension $E|K$. Therefore we obtain a function

$$h_{\widehat{\mathcal{L}}} : X(\overline{\mathbb{Q}}) \to \mathbb{R}.$$

Clearly,

$$h_{\widehat{\mathcal{L}_1} \otimes \widehat{\mathcal{L}_2}} = h_{\widehat{\mathcal{L}}_1} + h_{\widehat{\mathcal{L}}_2}, \quad h_{\widehat{\mathcal{L}}^\vee} = -h_{\widehat{\mathcal{L}}}.$$

Degrees and heights also respect pull-backs: let $X \xrightarrow{f} Y$ be a morphism between projective K-varieties, compatible with the given $\mathfrak{o}_K$-models, and let $\widehat{\mathcal{L}}$ be on Y. Then the pull-back $f^* \widehat{\mathcal{L}}$ makes sense and

$$h_{f^* \widehat{\mathcal{L}}} = h_{\widehat{\mathcal{L}}} \circ f$$

as functions on $X(\overline{\mathbb{Q}})$. We won't go into the details.

Example 2.2. Consider $X = \mathbb{P}^N$, $\mathcal{L} = \mathcal{O}(1)$ endowed with the standard $\mathbb{Z}$-integral structure + Fubini–Study metric at ∞. Write $X_0, \ldots, X_N$ for the standard global sections of $\mathcal{O}(1)$. Recall that the Fubini–Study metric on $\mathcal{O}(1)$ is

$$\|(a_0 X_0 + \cdots + a_N X_N)(x)\|_{v,x} = \frac{\left|\sum_{i=0}^N a_i x_i\right|_v}{\sqrt{\sum_{i=0}^N |x_i|_v^2}}, \quad v \mid \infty,$$

where $a_0, \ldots, a_N \in K_v$ and $x = (x_0 : \cdots : x_N)$.

Fix a number field K and Let $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N(K)$. It extends to $\mathbb{P}^N(\mathfrak{o}_K)$. For $v \nmid \infty$, its image in $\mathbb{P}^N(\mathfrak{o}_{K_v})$ is easily described as $(x_0/t : \cdots : x_N/t)$, where $t \in K_v$ satisfies $|t|_v = \max_{0 \leq i \leq N} |x_i|_v$.

Without loss of generality, suppose $x_0 \neq 0$. Take $\ell := X_0$ and use the observation above to compute the degree:

$$\|\ell\|_v = \begin{cases} \dfrac{|x_0|_v}{\sqrt{\sum_{i=0}^N |x_i|_v^2}}, & v \mid \infty \\ \dfrac{|x_0|_v}{\max_{0 \leq i \leq N} |x_i|_v} & v \nmid \infty. \end{cases}$$

The terms $|x_0|_v$ drop out after taking $\prod_v$, by the product formula. We obtain

$$h_{\widehat{\mathcal{L}}}(x) = \frac{1}{[K : \mathbb{Q}]} \left(\sum_{v \mid \infty} \log \sqrt{\sum_{i=0}^N |x_i|_v^2} + \sum_{v \nmid \infty} \log \max_{0 \leq i \leq N} |x_i|_v \right).$$

In what follows, $O(1)$ means the set of bounded functions on $X(\overline{\mathbb{Q}})$.

Proposition 2.3. *Modulo $O(1)$, the function $h_{\widehat{\mathcal{L}}}$ depends only on the line bundle $\mathcal{L}$ over X.*

Proof. We want to compare $\exp h_{\widehat{\mathcal{L}}}(x)$ for two families of v-adic metrics $\|\cdot\|_v$, $\|\cdot\|'_v$ defining metrized line bundles on X. One should bound the other by multiplicative constants.

By compactness, there exists $C_\infty > 0$ such that for each embedding $\iota : K \hookrightarrow$, the corresponding Hermitian forms satisfy

$$\|\cdot\|'_\iota \leq C_\infty \|\cdot\|_\iota.$$

For the non-Archimedean case, suppose that the integral structures for $\|\cdot\|$ and $\|\cdot\|'$ give rise to "lattices" $\mathcal{L}^\circ$, $\mathcal{L}^\dagger$ in $\mathcal{L}$, respectively. Then there is $N \in \mathbb{Z}_{\geq 1}$ such that

$$N\mathcal{L}^\circ \subset \mathcal{L}^\dagger.$$

This is a general property for K-schemes of finite type, and can be checked on affine open subsets. It entails that for every place $v \nmid \infty$ of K,

$$\|\cdot\|'_v \leq |N|_v^{-1} \|\cdot\|_v.$$

Note that $|N|_v = 1$ for almost all v. This gives the required bound of $(\prod_v \|\cdot\|')^{1/[K:\mathbb{Q}]}$ by $(\prod_v \|\cdot\|_v)^{1/[K:\mathbb{Q}]}$.

Moreover, the implied constants in the inequalities must be uniform when K is replaced by some finite extension E. This is guaranteed by the power $1/[K : \mathbb{Q}]$ and the equality $[E : K] = \sum_{w|v} [E_w : K_v]$, etc. $\square$

We will denote this $O(1)$-coset as $h_{\mathcal{L}}$.

Theorem 2.4 (Northcott property for heights). *If $\mathcal{L}$ is ample, then for all $d \in \mathbb{Z}_{\geq 1}$ and $B \in \mathbb{R}$, the set*

$$\left\{x \in X(L) : [L : K] \leq d,\ h_{\widehat{\mathcal{L}}}(x) \leq B\right\}$$

is finite. Here $\widehat{\mathcal{L}}$ is any metrization of $\mathcal{L}$.

Proof. We are free to alter the Hermitian and integral structures on $\mathcal{L}$. Upon replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ (thus $h_{\widehat{L}}$ by $mh_{\widehat{\mathcal{L}}}$) with $m \gg 0$, we may assume $\mathcal{L} = i^*\mathcal{O}(1)$ via a projective embedding $i : X \hookrightarrow \mathbb{P}^N$. The problem reduces to the case $X = \mathbb{P}^N$ and $\mathcal{L} = \mathcal{O}(1)$, with the standard integral structure and Fubini–Study metrics. Denote the corresponding height function as h.

Consider $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N(K)$. When $K = \mathbb{Q}$, we may further assume that $x_0, \ldots, x_N \in \mathbb{Z}$ are coprime. The formula for $h(x)$ reduces our assertion to showing that

$$\left\{\vec{x} = (x_1, \ldots, x_N) \in \mathbb{Z}^N : \vec{x} \neq 0,\ \log\sqrt{\sum_{i=0}^{N} x_i^2} \leq B\right\}$$

is a finite set. This is trivial.

For general K, let $\{x^{(1)}, \ldots, x^{(d)}\}$ be the Galois orbit of x in $\mathbb{P}^N(\overline{\mathbb{Q}})$, so they have the same height and $d \leq [K : \mathbb{Q}]$. We may identify x with the $\mathbb{Q}$-point $\{x^{(1)}, \ldots, x^{(d)}\}$ of the variety $\mathrm{Sym}^d\, \mathbb{P}^N$. We are reduced to the previous case using an explicit embedding $\phi : \mathrm{Sym}^d\, \mathbb{P}^N \hookrightarrow \mathbb{P}^M$ over $\mathbb{Q}$; one possible reference for this classical construction is [GKZ94, Chapter 4, §2.A] (Chow embeddings). The key is a bound of the form $h(\phi(x)) \leq ah(x) + b$, where a, b depend solely on N, d. □

3 Metrics with logarithmic singularities

We shall begin with the complex story. Let $\overline{X}$ be a compact complex analytic space, $Y \subset \overline{X}$ a closed subspace and $X := \overline{X} \smallsetminus Y$.

Definition 3.1. A log-distance to Y is a function $\rho : X \to \mathbb{R}_{>0}$ such that, locally around each $y \in Y$, we have

$$\rho \approx -\log \sum_{i=1}^{r} |f_i|^2$$

where $f_1 = \cdots = f_r = 0$ is a system of local equations for Y around y.

Now let $\mathcal{L}$ be a line bundle over $\overline{X}$, equipped with an Hermitian metric $\|\cdot\|$ over X. On the other hand, we can always equip $\mathcal{L}$ with some $\|\cdot\|_0$ over all X, say by local trivializations and partition of unity.

Definition 3.2. We say that $(\mathcal{L}, \|\cdot\|)$ has logarithmic singularity along the boundary Y if there exists an Hermitian metric $\|\cdot\|_0$ on $\mathcal{L}$ over all $\overline{X}(\mathbb{C})$, such that

$$\max\left\{\frac{\|\cdot\|_0}{\|\cdot\|}, \frac{\|\cdot\|}{\|\cdot\|_0}\right\} = O(\rho^r)$$

near $Y(\mathbb{C})$, where $r \gg 0$ and ρ is some log-distance to Y.

It is routine but somewhat laborious to check that the notion above is invariant under pullbacks, and can be checked "upstairs" under proper morphisms. In particular, it is birationally invariant.

Now revert to the global setting. Let K be a number field.

Definition 3.3. Suppose that $\overline{X}$ is the base change to K of a proper $\mathfrak{o}_K$-scheme $\overline{\mathcal{X}}$, and $\overline{X}$ is projective. Let $\mathcal{L}$ be a line bundle over $\overline{X}$ with an $\mathfrak{o}_K$-model. Consider an open K-subscheme $X \subset \overline{X}$. Suppose that we are given Hermitian metrics $\|\cdot\|_\iota$ on $\mathcal{L}_{\mathbb{C},\iota}$ for each $\iota : K \hookrightarrow \mathbb{C}$, of logarithmic singularity along $Y := \overline{X} \smallsetminus X$. These data allow us to define

$$h_{\mathcal{L},\|\cdot\|}(x) := \deg x^*\mathcal{L}.$$

Indeed, in computing $\deg x^*\mathcal{L}$, the integral structure is only used at non-Archimedean places, and the Hermitian metrics involve just $x : \operatorname{Spec} K \to X$.

Theorem 3.4. *In the situation above, suppose $\mathcal{L}$ is ample. Then for all $d \in \mathbb{Z}_{\geq 1}$ and $B \in \mathbb{R}$, the set*

$$\left\{x \in X(L) : [L : K] \leq d,\ h_{\mathcal{L},\|\cdot\|}(x) \leq B\right\}$$

is finite.

Proof. To simply notation, we shall assume that there is only one Archimedean place for K. Take an Hermitian metric $\|\cdot\|_0$ for $\mathcal{L}_\mathbb{C}$ defined over $\overline{X}_\mathbb{C}$. Fix a projective embedding $\overline{X} \hookrightarrow \mathbb{P}^N$. Let $D \subset \mathbb{P}^N$ be a hypersurface containing Y. For every $\epsilon > 0$, take an Hermitian metric $\|\cdot\|_{D,\epsilon}$ on $\overline{X}_\mathbb{C} \smallsetminus D_\mathbb{C}$ with the following asymptotic behavior near $D_\mathbb{C}$

$$\|\cdot\|_{D,\epsilon} \approx \|\cdot\|_0 \cdot |f|^{-\epsilon}$$

where f is a local equation for D. This is always possible. As polynomials dominate logarithms, we have

$$\|\cdot\| \ll_\epsilon \|\cdot\|_{D,\epsilon} \quad \text{on } X_\mathbb{C} \smallsetminus D.$$

Hence $h_{\mathcal{L},\|\cdot\|} \gg_\epsilon h_{\mathcal{L},\|\cdot\|_{D,\epsilon}}$ on $(X \smallsetminus D)(\overline{\mathbb{Q}})$. Now take $n \in \mathbb{Z}_{\geq 1}$ and $\epsilon := \frac{1}{n}$. Note that $\|\cdot\|_0^n \cdot |f|^{-1}$ is an Hermitian metric for $\mathcal{L}^{\otimes n}(-D)$ over the whole $\overline{X}_\mathbb{C}$. Over $(\overline{X} \smallsetminus D)(\overline{\mathbb{Q}})$ we have

$$n \cdot h_{\mathcal{L},\|\cdot\|_{D,1/n}} \approx h_{\mathcal{L}^{\otimes n},\|\cdot\|_0^n \cdot |f|^{-1}} = h_{\mathcal{L}^{\otimes n}(-D)} + \text{bdd fcn on } \overline{X}(\overline{\mathbb{Q}});$$

Hence over $(X \smallsetminus D)(\mathbb{Q})$ we have

$$h_{\mathcal{L},\|\cdot\|} \gg h_{\mathcal{L},\|\cdot\|_{D,1/n}} \geq \frac{1}{n} h_{\mathcal{L}^{\otimes n}(-D)} + \text{bdd fcn on } \overline{X}(\overline{\mathbb{Q}}).$$

Take $n \gg 0$ so that $\mathcal{L}^{\otimes n}(-D)$ is ample and $\left\{x \in \overline{X}(\overline{\mathbb{Q}}) : h_{\mathcal{L}^{\otimes n}(-D)}(x) \leq \text{const}\right\}$ is finite. Then $\left\{x \in (X \smallsetminus D)(\overline{\mathbb{Q}}) : h_{\mathcal{L},\|\cdot\|}(x) \leq \text{const}\right\}$ is finite as well. Vary D to conclude. □

4 Moduli of abelian schemes and their compactifications

Definition 4.1. Let S be a normal scheme. A semi-abelian S-scheme is a smooth separated commutative S-group scheme G, which is geometrically connected, and that over every point s, its fiber fits into a short exact sequence of groups over the residual field

$$1 \to T_s \to G_s \to A_s \to 1, \quad T_s : \text{torus}, \; A_s : \text{abelian variety}.$$

For a semi-abelian S-scheme G of relative dimension g and unit section e, we have the following line bundle on S

$$\omega_G := e^* \Omega^g_{G|S}.$$

Now consider an abelian S-scheme A. When $S = \operatorname{Spec} \mathbb{C}$, there is a canonical Hermitian metric on ω_A, namely

$$(\alpha|\beta) := \sqrt{-1}^{g^2} \int_{A(\mathbb{C})} \alpha \wedge \overline{\beta}.$$

The next step is to vary A. Take S to be the moduli stack $\mathcal{A}_g$ over $\mathbb{Z}$ of principally polarized abelian schemes of dimension g; what matters for us is just its coarse moduli space. It is the base of the universal principally polarized abelian scheme $A_{\text{univ}} \to \mathcal{A}_g$. We obtain the Hodge line bundle ω over $\mathcal{A}_g$, whose fiber over an element of $\mathcal{A}_g(S)$, identified with an abelian scheme $A \to S$, is just ω_A.

Remark 4.2. To obtain moduli spaces as bona fide schemes, one can rigidify A by adding level structures $(\mathbb{Z}/n\mathbb{Z})^{2g}_S \xrightarrow{\sim} A[n]$ with $n \geq 3$. Cf. the Michigan notes.

We also need some compactifications of $\mathcal{A}_g$ over $\mathbb{Z}$.

1. The *minimal compactification* $\mathcal{A}_g \hookrightarrow \mathcal{A}^*_g$ of Baily–Borel. Its definition over $\mathbb{C}$ is relatively easy: simply take the normal projective embedding of $\mathcal{A}_g$ induced by the graded algebra of Siegel modular forms. This compactification is rarely smooth.

2. The *toroidal compactifications* $\mathcal{A}_g \hookrightarrow \overline{\mathcal{A}_g}$. They are defined in terms of rational polyhedral combinatorial data. Toroidal compactifications are not canonical: only the tower thereof is a canonical object. Each $\overline{\mathcal{A}_g}$ is proper and smooth over $\operatorname{Spec} \mathbb{Z}$, and there is a semi-abelian scheme $G \to \overline{\mathcal{A}_g}$ extending $A_{\text{univ}} \to \mathcal{A}_g$. Consequently, ω extends to a line bundle over $\overline{\mathcal{A}_g}$, namely $\omega_{G|\overline{\mathcal{A}_g}}$.

Every toroidal compactification dominates the minimal one via a proper surjection $\overline{\mathcal{A}_g} \xrightarrow{\pi} \mathcal{A}^*_g$. In [FC90, V.2.3 Theorem] is defined a very ample line bundle $\mathcal{L}$ on $\mathcal{A}^*_g$, obtained as the descent of $\omega^{\otimes m}$ along π for large $m \in \mathbb{Z}_{\geq 1}$.

Also recall the following

Definition 4.3. Let R be a Dedekind ring and $K = \operatorname{Frac}(R)$. An abelian variety A over K is said to have *semi-stable reduction* [1] if the identity connected component $\mathfrak{A}^\circ$ of its Néron model $\mathfrak{A}$ is a semi-abelian R-scheme.

Remark 4.4. The arithmetic theory of toroidal compactifications of $\mathcal{A}_g$ was not yet available when Faltings wrote his proof. Instead, he worked over $\mathbb{C}$ and employed other tricks such as Gabber's Lemma to prove the main finiteness theorem, see [Del85]. The argument below will follow the later approach of [FC90, V.4], which is more straightforward.

[1] Some authors call it semi-abelian reduction.

5 Comparison of heights

Let A be an abelian variety of dimension g over a number field F. By using the connected Néron model $\mathfrak{A}^\circ$, we extend ω_A to an invertible sheaf over $\operatorname{Spec} \mathfrak{o}_F$. On the other hand, for every embedding $\iota : F \hookrightarrow \mathbb{C}$ we have the canonical Hermitian form $(\cdot|\cdot)_\iota$ as before. Being canonical, it is evidently conjugation-invariant. All in all, we have made ω_A into a metrized line bundle over $\operatorname{Spec} \mathfrak{o}_F$.

Definition 5.1. The modular height of A is

$$h_{\mathrm{mod}}(A) := \frac{1}{[F:\mathbb{Q}]} \deg \omega_A.$$

For principally polarized A, we are going to define another height $h_{\mathrm{geom}}(A)$. In order to apply it, the following result will be crucial.

Theorem 5.2. *Consider the line bundle $\mathcal{L}$ over $\mathcal{A}^*_{g,\mathbb{C}}$. The Hermitian form $(\cdot|\cdot)$ on $\mathcal{L}|_{\mathcal{A}_{g,\mathbb{C}}}$ has logarithmic singularity along the boundary.*

Proof. We only discuss the simplest case $g = 1$. Then $\mathcal{A}_g$ equals $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H} =: Y(1)$, and the $\mathbb{C}$-analytifications of $\mathcal{A}^*_g$ and $\overline{\mathcal{A}_g}$ both equal $X(1) = Y(1) \sqcup \{\infty\}$. It suffices to look at the vicinity of the cusp ∞, namely

$$\begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix} \backslash \{\tau \in \mathbb{C} : \Im(\tau) > c\}, \quad c \gg 0.$$

Each τ parameterizes the complex torus $E_\tau := \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$ with its unique principal polarization. Let z be the standard coordinate function on E_τ. The line bundle ω is trivialized in this neighborhood of ∞ by the $\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ & 1 \end{smallmatrix}\right)$-invariant section $\mathrm{d}z$. Now compute the Hermitian form: we have

$$(\mathrm{d}z|\,\mathrm{d}z) = \sqrt{-1} \int_{E_\tau} \mathrm{d}z \wedge \overline{\mathrm{d}z} = 2\Im(\tau).$$

The local coordinate function around ∞ is $q := e^{2\pi\sqrt{-1}\tau}$. Note that $\log|q| = -2\pi\Im\tau$. The logarithmic singular behavior is thus evident.

For general g, the logarithmic singularity is checked "upstairs" on the toroidal compactifications $\overline{\mathcal{A}_g}$. This requires some knowledge about the local structure of toroidal compactifications over $\mathbb{C}$. For complete arguments, see [FC90, V.4.5 Proposition]. ☐

Observe that Hermitian form $(\cdot|\cdot)$ on $\omega|_{\mathcal{A}_{g,\mathbb{C}}}$ passes to $\mathcal{L}|_{\mathcal{A}_{g,\mathbb{C}}} \simeq \omega^{\otimes m}|_{\mathcal{A}_{g,\mathbb{C}}}$. This permits us to define the height $h_{\widehat{\mathcal{L}}} : \mathcal{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$ using that Hermitian form over $\mathcal{A}_g$ and the integral structure inherent in $\mathcal{L}$ and $\mathcal{A}^*_g$.

The $O(1)$-coset of the height function is independent of the integral structure and Hermitian metrics, but our choice will facilitate the comparison with h_{geom}.

Definition 5.3. Let A be principally polarized and identify it as an element of $\mathcal{A}_g(F)$. The geometric height[2] of A is

$$h_{\mathrm{geom}}(A) := \frac{1}{m} h_{\widehat{\mathcal{L}}}(A).$$

By construction, $h_{\mathrm{geom}}(A)$ is independent of m.

[2] Geometric by its invariance under finite extensions $K|F$. On the other hand, h_{mod} is sometimes known as the Faltings height.

Lemma 5.4. *Suppose that A is principally polarized and has semi-stable reduction. Then*

$$h_{\mathrm{geom}}(A) = h_{\mathrm{mod}}(A).$$

Proof. By the definition of semi-stable reduction, the connected Néron model $\mathfrak{A}^\circ$ is a semi-abelian $\mathfrak{o}_F$-scheme. From [FC90, IV.5.7 Theorem (5)] or [FC90, IV.5.1 Proposition] (in essence, the uniqueness of semi-abelian models) we infer the existence of

- a toroidal compactification $\overline{\mathcal{A}_g}$ of $\mathcal{A}_g$,
- a morphism $\mathrm{Spec}\, \mathfrak{o}_F \xrightarrow{f} \overline{\mathcal{A}_g}$ whose generic fiber is the morphism $\mathrm{Spec}\, F \to \mathcal{A}_g$ classifying A,

such that $\mathfrak{A}^\circ \to \mathrm{Spec}\, \mathfrak{o}_F$ is the pullback of the semi-abelian scheme $G \to \overline{\mathcal{A}_g}$. Summing up,

$$\begin{array}{ccccc} \mathcal{A}_g & \hookrightarrow & \mathcal{A}_g^* & \overset{\pi}{\twoheadleftarrow} & \overline{\mathcal{A}_g} \\ \uparrow & & \uparrow & \nearrow_{f} & \\ \mathrm{Spec}\, F & \longrightarrow & \mathrm{Spec}\, \mathfrak{o}_F & & \end{array} \qquad \text{commutes.}$$

Therefore $\mathcal{L}$ pulled back to $\mathrm{Spec}\, \mathfrak{o}_F$ equals $\omega^{\otimes m} = \pi^*\mathcal{L}$ pulled back from $\overline{\mathcal{A}_g}$ to $\mathrm{Spec}\, \mathfrak{o}_F$ as metrized line bundles. Remember that in defining $h_{\mathrm{mod}}(A)$ (resp. $h_{\mathrm{geom}}(A)$) the integral structure on ω_A (resp. on $\mathcal{L}$) is retrieved from $\mathfrak{A}^\circ \to \mathrm{Spec}\, \mathfrak{o}_F$ (resp. from $G \to \overline{\mathcal{A}_g}$). This compatibility suffices to conclude. □

Lemma 5.5. *Let $K|F$ be a finite extension of number fields. For all principally polarized abelian varieties $A = A_F$ over F of dimension g, we have $h_{\mathrm{mod}}(A_F) \geq h_{\mathrm{mod}}(A_K)$.*

Proof. The characterization of Néron models gives a morphism of abelian $\mathfrak{o}_K$-schemes

$$\mathfrak{A}_F \underset{\mathfrak{o}_F}{\times} \mathfrak{o}_K \to \mathfrak{A}_K,$$

inducing a canonical arrow between invertible $\mathfrak{o}_K$-modules

$$\Phi : \omega_{A_K} \to \omega_{A_F} \underset{\mathfrak{o}_F}{\otimes} \mathfrak{o}_K.$$

In turn, it induces an isomorphism on generic fibers, whence an isometry at each $v \mid \infty$. For each $v \nmid \infty$, we claim:

$$\|\ell\|_v \geq \|\Phi(\ell)\|_v, \quad \ell \in \omega_{A_K} \underset{\mathfrak{o}_K}{\otimes} K_v,$$

with respect to integral structures on ω_{A_K} and $\omega_{A_F} \underset{\mathfrak{o}_F}{\otimes} \mathfrak{o}_K$. To see this, simply take ℓ to be a generator of ω_{A_K}, so that the left hand side is 1 and the right hand side is ≤ 1 (being integral). Taking the sum of $-\log \|\cdot\|_v$, it follows that

$$[K:\mathbb{Q}]^{-1} \deg \omega_{A_K} \leq [K:\mathbb{Q}]^{-1} \deg \left(\omega_{A_F} \underset{\mathfrak{o}_F}{\otimes} \mathfrak{o}_K \right) = [F:\mathbb{Q}]^{-1} \deg \omega_{A_F}$$

as required. □

Theorem 5.6. *Let A be a principally polarized abelian variety over a number field F, we have $h_{\mathrm{mod}}(A) \geq h_{\mathrm{geom}}(A)$.*

Proof. There exists a finite extension $K|F$ such that A_K acquires semi-stable reduction. Then

$$h_{\mathrm{mod}}(A) \geq h_{\mathrm{mod}}(A_K) = h_{\mathrm{geom}}(A_K) = h_{\mathrm{geom}}(A),$$

the last equation being a general property of Arakelov heights. □

We deduce the Northcott property of h_{mod} as follows.

Corollary 5.7. *For every $g, d \in \mathbb{Z}_{\geq 1}$ and $B \in \mathbb{R}$, the set*

$$\left\{ \begin{array}{c|c} F : \textit{number field}, & [F:\mathbb{Q}] \leq d,\ \dim A = g \\ A : \textit{abelian variety over } F\ /\simeq & h_{\mathrm{mod}}(A) \leq B \\ \textit{principally polarized} & \end{array} \right\}$$

is finite.

Proof. This follows immediately from $h_{\mathrm{mod}}(A) \geq h_{\mathrm{geom}}(A)$ and the Northcott property for $h_{\widehat{\mathcal{L}}}$ (i.e. for h_{geom}), as latter has logarithmic singularities along $\mathcal{A}_g^* \setminus \mathcal{A}_g$. □

Remark 5.8. One can use Zarhin's trick to remove the condition of principal polarization.

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Last update: November 23, 2018