A user's guide to the trace formula for covering groups

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References

1. Arthur's papers.
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3. *La formule des traces pour les revêtements de groupes réductifs connexes. I.*
   Le développement géométrique fin (arXiv:1004.4011)
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5. *La formule des traces pour les revêtements de groupes réductifs connexes. III.*
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6. *La formule des traces pour les revêtements de groupes réductifs connexes. IV.*
   Distributions invariantes (in prepartation)

Arthur-Selberg trace formula

- F: number field, $\mathbb{A}$ its ring of adèles,
- G: a connected reductive group over F,
- $G(\mathbb{A})^1 := \mathrm{Ker}(H_G)$ where $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$ is the Harish-Chandra homomorphism,
- R: right regular representation of $\tilde{G}$ on $L^2(G(F)\backslash G(\mathbb{A})^1)$,
- $f \in C_c^\infty(G(\mathbb{A}))$,
- K_G: the kernel of $R(f)$, $k(x) := K_G(x, x)$ for $x \in G(\mathbb{A})$.

The Arthur-Selberg trace formula calculates a truncated integral of $k(x)$ over $G(F)\backslash G(\mathbb{A})^1$:

$$\text{(geometric expansion)} = J(f) = \text{(spectral expansion)}.$$

Roughly speaking:

- the geometric side: distributions on $G(\mathbb{A})^1$ indexed by rational conjugacy classes (eg. orbital integrals),
- the spectral side: distributions on $G(\mathbb{A})^1$ indexed by automorphic representations (eg. characters).

Some of the applications

- Base change and Jacquet-Langlands correspondence for $\mathrm{GL}(n)$.
- Endoscopic classification of representations of classical groups.
- Formula for the trace of Hecke operators.
- Various results in local harmonic analysis (character identities, etc.)
- Applications in analytic number theory.

In each case, it is crucial to have some refined versions of this trace formula. Examples: invariant trace formula, stable trace formula.

Covers of connected reductive groups: the local case

Consider central extensions of locally compact groups as follows.

The local setting

F: local field, G: connected reductive F-group.

$$1 \to \mathbf{N} \to \tilde{G} \to G(F) \to 1,$$

where $\mathbf{N}$ is finite abelian.

The global setting

F: global field, $\mathbb{A}$ its ring of adèles, G: connected reductive F-group.

$$1 \to \mathbf{N} \to \tilde{G} \to G(\mathbb{A}) \to 1.$$

Examples

- $G = \mathrm{Sp}(2n)$: A. Weil (1964) $\Rightarrow$ representation-theoretic interpretation of Siegel modular forms of half-integral weight.
- $G = \mathrm{SL}(2)$ or $\mathrm{GL}(2)$: Shimura (1973), Kubota.
- G split, simple and simply connected: R. Steinberg (1962), H. Matsumoto (1969) constructed the *universal central extension* of $G(F)$ – related to algebraic K-theory.
- $G = \mathrm{GL}_n$: metaplectic correspondence (Flicker, Kazhdan, Patterson,..., $\geq$ 1980).
- G arbitrary: Deligne and Brylinski (2001) classified their $\mathbf{K}_2$-extensions.

Genuine representations

In harmonic analysis, on may assume $\mathbf{N} = \mu_m := \{z \in \mathbb{C}^\times : z^m = 1\}$ for some m.

- It suffices to study the representations π of $\tilde{G}$ which are genuine, i.e. $\pi(\epsilon\tilde{x}) = \epsilon\pi(\tilde{x})$ for all $\epsilon \in \mu_m$.
- Test functions: it suffices to consider $\pi(f)$ with antigenuine $f \in C_c^\infty(\tilde{G})$, i.e. $f(\epsilon\tilde{x}) = \epsilon^{-1} f(\tilde{x})$.

Justification: given a smooth irrep π of $\tilde{G}$, let $\omega : \mathbf{N} \to \mathbb{C}^\times$ be its central character on $\mathbb{N}$. Then it suffices to study the push-forward of $1 \to \mathbf{N} \to \tilde{G} \to G(F) \to 1$ by ω.

Constraints on covers

The class of such extensions under consideration should be:

- stable under push-forward by any homomorphism $\mu_m \to \mu_{m'}$;
- stable under passage to Levi subgroups (*philosophy of cusp forms*);
- when F is global,
  - $\exists$ splitting $G(F) \hookrightarrow \tilde{G}$ ($\Rightarrow$ spectral decomposition, see [MW]),
  - $\exists$ splittings over hyperspecial subgroups $G(\mathfrak{o}_v)$ at almost all v such that $\prod_v G(\mathfrak{o}_v) \hookrightarrow \tilde{G}$ is continuous; here we fix an integral model of G;
  - (continued) the corresponding antigenuine spherical Hecke algebra at v must be commutative ($\Rightarrow \bigotimes$-decomposition of smooth irreps)

Existence of canonical splittings over unipotent subgroups: automatic. $\Rightarrow$ notions of constant terms and Jacquet functors.
These conditions are satisfied by the $\mathbf{K}_2$-extensions of Brylinski-Deligne.

Desiderata

Goal: establish the Arthur-Selberg trace formula for a large class of covers. Fix a minimal Levi M_0 and set $\mathcal{L}(M_0) := \{\text{Levi containing } M_0\}$

- The coarse trace formula

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f) = J(f) = \sum_{\chi} J_{\chi}(f).$$

- Refined trace formula, in terms of weighted characters and weighted orbital integrals:

$$\sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \sum_{\gamma \in \Gamma(\tilde{L}^1, V)} a^{\tilde{L}}(\tilde{\gamma}) J_{\tilde{L}}(\tilde{\gamma}, f_V) = J(f)$$

$$= \sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \int_{\Pi_-(\tilde{L}^1, V)} a^{\tilde{L}}(\pi) J_{\tilde{L}}(\pi, 0, f) d\pi.$$

- The invariant trace formula

$$\sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \sum_{\gamma \in \Gamma(\tilde{L}^1, V)} a^{\tilde{L}}(\tilde{\gamma}) I_{\tilde{L}}(\tilde{\gamma}, f_V) = I(f)$$

$$= \sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \int_{\Pi_-(\tilde{L}^1, V)} a^{\tilde{L}}(\pi) I_{\tilde{L}}(\pi, 0, f) d\pi.$$

where the $I_{\tilde{L}}(\cdots)$ are invariant distributions. For $L = G$, we get the usual orbital integrals and characters.

- Simple trace formula: for suitable choice of f, only the terms with $L = G$ survive.
- Long-term goal (for some special $\tilde{G}$): stabilization $\Rightarrow$ rewrite everything in terms of stable distributions on certain linear reductive groups.

Coarse trace formula

Let $\mathbf{p} : \tilde{G} \to G(\mathbb{A})$ be a cover, $\mathrm{Ker}(\mathbf{p}) = \mu_m$.

- $\tilde{G}^1 := \mathrm{Ker}(H_G \circ \mathbf{p})$ where $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$ is the Harish-Chandra homomorphism.
- R: right regular representation of $\tilde{G}$ on $L^2(G(F)\backslash\tilde{G}^1)$,
- $f \in C_c^\infty(\tilde{G})$ antigenuine,
- K_G: the kernel of $R(f)$, $k(x) := K_G(\tilde{x}, \tilde{x})$ for $x \in G(\mathbb{A})$, $\tilde{x} \in \mathbf{p}^{-1}(x)$,
- for any parabolic $P = MU$, R_P the right regular representation on $L^2(U(\mathbb{A})M(F)\backslash\tilde{G}^1)$ and K_P its kernel.

Fix minimal Levi M_0 and maximal compact $K \subset G(\mathbb{A})$ in good relative position. Set $\tilde{K} := \mathbf{p}^{-1}(K)$, $\mathfrak{a}_0 := \mathfrak{a}_{M_0}$.

Fix $P_0 \in \mathcal{P}(M_0)$. For $T \in \mathfrak{a}_0$, define the truncated kernel à la Arthur

$$k^T(x) := \sum_{P \supset P_0} (-1)^{\dim A_P/A_G} \sum_{\delta \in P(F)\backslash G(F)} K_P(\delta\tilde{x}, \delta\tilde{x}) \hat{\tau}_P(H_P(\delta x) - T).$$

Theorem

For T highly regular, $k^T(x)$ is integrable over $G(F)\backslash\tilde{G}^1$. There is an identity of absolutely convergent integrals

$$\sum_{\mathfrak{o}} J^T_{\mathfrak{o}}(f) = J^T(f) = \sum_{\chi} J^T_{\chi}(f).$$

Everything in sight is polynomial in T.

- Spectral side: χ ranges over *cuspidal data* (M, σ), where $M \supset M_0$ is a Levi subgroup, σ is a cuspidal automorphic representation of $\tilde{M}$ inside $L^2(M(F)\backslash\tilde{M}^1)$.
- Geometric side: $\mathfrak{o}$ ranges over semisimple classes in $G(F)$. The unipotent term $J^T_{\text{unip}}(f)$ corresponds to 1.

About the proof

- Combinatorics: the same as the case of reductive groups (Arthur),
- Spectral decomposition: included in Moeglin-Waldspurger,
- Geometric side: the same as in the case of reductive groups – we only look at conjugacy classes in $G(F)$.

Refinement

Let $T_0 \in \mathfrak{a}_0$ be the canonical element (depending on K) defined by Arthur. Then

$$J(f) := J^{T_0}(f)$$
$$J_\chi(f) := J_\chi^{T_0}(f)$$
$$J_\mathfrak{o}(f) := J_\mathfrak{o}^{T_0}(f)$$

The problem is to find explicit formulas for them.

Goal

1. Express $J_\chi(f)$, $J_\mathfrak{o}(f)$ in terms of weighted orbital integrals and weighted characters (local objects).
2. Isolate global and local information.

Descend to the unipotent case

Idea: get rid of the cover on the geometric side.
For $x, y \in G(\mathbb{A})$ with liftings $\tilde{x}, \tilde{y} \in \tilde{G}$, set

$$[x, y] := \tilde{x}^{-1}\tilde{y}^{-1}\tilde{x}\tilde{y}.$$

Let $\sigma \in G(F)$ be semisimple, $G_\sigma := Z_G(\sigma)^\circ$. Then $[\cdot, \sigma]$ defines a homomorphism $G_\sigma(\mathbb{A}) \to \mu_m$.

Principle

Let $\mathfrak{o}$ be the $G(F)$-orbit containing σ. Reduce $J_{\mathfrak{o}}^{\tilde{G}}(f)$ to $J_{\text{unip}}^{G_\sigma,[\cdot,\sigma]}$, the unipotent term of the trace formula of G_σ twisted by the character $[\cdot, \sigma]$. (More precisely, some Levi subgroups of G_σ may appear...)

Remark: we have Jordan decomposition on covers!

Example: descent of orbital integrals

The same formalism about $[\cdot, \sigma]$ applies to the local case. Let F be a local field, $\mathbf{p} : \tilde{G} \to G(F)$ a cover and $f \in C_c^\infty(\tilde{G})$ antigenuine.

Theorem

$\exists f^\flat \in C_c^\infty(\mathfrak{g}_\sigma(F))$ such that $\forall \tilde{\gamma} = \sigma \exp(X)$ with $X \in \mathfrak{g}_\sigma(F)$ sufficiently close to 0, we have

$$|D^G(\gamma)|^{\frac{1}{2}} O_{\tilde{\gamma}}^{\tilde{G}}(f) = |D^{G_\sigma}(X)|^{\frac{1}{2}} O_X^{G_\sigma, [\cdot,\sigma]}(f^\flat)$$

where D^G and D^{G_σ} are the Weyl discriminants on G and $\mathfrak{g}_\sigma$, respectively.

It's tempting to remove $[\cdot, \sigma]$ and replace $G_\sigma(F)$ by $\mathrm{Ker}([\cdot, \sigma])|_{G_\sigma(F)}$. However the latter group is less manageable.

Basic ideas for the refined geometric expansion

1. Reduce to the study of $J_{\text{unip}}^{G_\sigma,[\cdot,\sigma]}(f)$.
2. Express $J_{\text{unip}}^{G_\sigma,[\cdot,\sigma]}(f)$ in terms of weighted unipotent orbital integrals twisted by $[\cdot,\sigma]$ (adapt Arthur's arguments).
3. Define weighted orbital integrals on $\tilde{G}$ and deduce analogous descent formulas.
4. Compare the descent formulas to express $\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f)$ in terms of weighted orbital integrals on $\tilde{G}$.

The result will be expressed in terms of good weighted orbital integrals $J_{\tilde{M}_S}(\tilde{\gamma}_S, f_S)$, where

- S: a sufficiently large finite set of places containing the archimedean and the ramified ones, depending on $\mathrm{Supp}(f)$;
- $\tilde{M}_S := \mathbf{p}^{-1}(M(F_S))$;
- $f = f_S f_{K^S}$, where f_{K^S} is the unit in the antigenuine spherical Hecke algebra of $\tilde{G}^S$ w.r.t. $K^S := \prod_{v \notin S} K_v$;
- $\tilde{\gamma}_S$: conjugacy class in $\tilde{M}_S$ which is good, i.e. $\tilde{x}\tilde{\gamma}_S = \tilde{\gamma}_S\tilde{x}$ iff $x\gamma_S = \gamma_S x$, where $\mathbf{p}(\tilde{*}) = *$.

When $M = G$, we get the usual orbital integrals.

Regular semisimple weighted orbital integrals

Let $S = \{v\}$, $\tilde{\gamma} \in \tilde{M}_v$ be semisimple, regular and good; in $\tilde{G}_v$ ($M \subset G$: Levi). Then

$$J_{\tilde{M}_v}(\tilde{\gamma}, f_v) = |D^M(\gamma)|^{\frac{1}{2}} \int_{M_\gamma(F_v)\backslash G(F_v)} f_v(x^{-1}\tilde{\gamma}x) v_M(x) dx.$$

- $v_M : M(F)\backslash G(F)/K_v \to \mathbb{R}_{\geq 0}$ is the volume for some convex polytope in $\mathfrak{a}_M^G$;
- $J_{\tilde{M}_v}(\tilde{\gamma}, f_v)$ depends on
  - the choice of K_v,
  - a Haar measure on the $\mathbb{R}$-v.s. $\mathfrak{a}_M^G$,
  - the ratio of the Haar measures on $G(F)$ and $G_\gamma(F)$.
- For general S: can be reduced to $S = \{v\}$ via Arthur's splitting formula.
- For general $\tilde{\gamma}$: defined by a limiting process.

Let us return to the refined geometric expansion. We have:

$$J(f) = \sum_{M \in \mathcal{L}(M_0)} \frac{|W_0^M|}{|W_0^G|} \sum_{\substack{\gamma \in (M(F))_{M,S}^{K,\text{good}} \\ \gamma \rightsquigarrow \widetilde{\gamma_S}}} a^{\tilde{M}}(S, \dot{\widetilde{\gamma_S}}) J_{\tilde{M}_S}(\dot{\widetilde{\gamma_S}}, f).$$

- S: depending on $\mathrm{Supp}(f)$.
- $(M(F))_{M,S}$: the set of (M,S)-equivalence classes defined by Arthur.
- $(\cdots)_{M,S}^{K,\text{good}}$: the classes γ admitting a representative whose components outside S are in K^S, and whose local component $\widetilde{\gamma_S}$ in $\tilde{M}_S$ is good (using $\mathbf{p}^{-1}(M(F_S) \times K^S) = \tilde{M}_S \times K^S$).
- The correspondence $\gamma \rightsquigarrow \widetilde{\gamma_S}$: described as above.

One difficulty

Arthur's proof can be adapted to prove the refined geometric expansion. But it requires new ideas as well.

- In the course of proof, one has to show that $a^{\tilde{M}}(S, \dot{\widetilde{\gamma_S}}) J_{\tilde{M}_S}(\dot{\widetilde{\gamma_S}}, f)$ behaves well with respect to the correspondence $\gamma \rightsquigarrow \widetilde{\gamma_S}$.
- Unlike the case treated by Arthur, the character $[\cdot, \sigma]$ intervenes and we need a somewhat technical result of "transport of structure" for $a^{\tilde{M}}(S, \dot{\widetilde{\gamma_S}})$ and $J_{\tilde{M}_S}(\dot{\widetilde{\gamma_S}}, f)$.

Compression of coefficients

Note: the set of places S depends on the support of f.
Goal: we want an expression depending only on a set V containing $\{v : v|\infty\}$ and the ramified places, and we use test functions of the form $f = f_V f_{K^V}$, where f_{K^V} is the unit of the antigenuine spherical Hecke algebra w.r.t. $(\tilde{G}^V, K^V)$.

Theorem

For V as above, there are

- *a set $\Gamma(\tilde{L}^1, V)$ of good conjugacy classes in $\tilde{L}_V$;*
- *coefficients $a^{\tilde{L}}(\tilde{\gamma})$ for $\tilde{\gamma} \in \Gamma(\tilde{L}^1, V)$,*

such that

$$J(f) = \sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \sum_{\gamma \in \Gamma(\tilde{L}^1, V)} a^{\tilde{L}}(\tilde{\gamma}) J_{\tilde{L}}(\tilde{\gamma}, f_V).$$

The compressed coefficients $a^{\tilde{L}}(\tilde{\gamma})$ are defined in terms of

1. the global coefficients $a^{\tilde{M}}(S, \cdot)$, for various $M \subset L$ and $S \supset V$ large enough;
2. the unramified weighted orbital integrals $r^L_M(\cdot)$, i.e. the weighted orbital integrals of the unit in the genuine unramified spherical Hecke algebra $\tilde{G}^V_S$ w.r.t. K^V_S.

The refined spectral side

Suppose f to be left and right $\tilde{K}$-finite from now on. Then

$$J_\chi(f) = \sum_{M \in \mathcal{L}(M_0)} \sum_{\pi \in \Pi(\tilde{M}^1)} \sum_{L \in \mathcal{L}(M)} \sum_{s \in W^L(M)_{\mathrm{reg}}} \frac{|W_0^M|}{|W_0^G|} \cdot$$
$$\cdot \, |\det(s-1|\mathfrak{a}_M^L)|^{-1} \int_{i(\mathfrak{a}_L^G)^*} \mathrm{tr}(\mathcal{M}_L(\tilde{P},\lambda) M_{P|P}(s,0) I_{\tilde{P}}^{\tilde{G}}(\lambda, f)_{\chi,\pi}) \mathrm{d}\lambda.$$

- $\Pi(\tilde{M}^1)$: the set of unitary irreps of $\tilde{M}^1$ up to equivalence,
- $P \in \mathcal{P}(M)$ arbitrary,
- $M_{P|P}(s,0)$: global intertwining operators,
- $\mathcal{M}_L(\tilde{P},\lambda)$: an operator defined by a (G,L)-family arising from intertwining operators,
- $I_{\tilde{P}}^{\tilde{G}}(\cdots)$: unitary parabolic induction.

Local ingredients of the proof

Mostly Harish-Chandra's theory:

- local intertwining operators,
- c-functions, μ-functions,
- Plancherel formula,
- normalization of local intertwining operators:
  - Archimedean case: juggling with Γ-functions,
  - non-archimedean case: adapt Langlands' proof,
  - "unramified" case: need some theory of unramified genuine principal series, cf. [McNamara].

Global ingredients of the proof

In view of Moeglin-Waldspurger, one can simply copy Arthur's arguments.

Global weighted characters

One can define

- a set $\Pi_{\mathsf{disc},-}(\tilde{M}^1)$ of "genuine discrete parameters" for each Levi M;
- the discrete coefficients $a^{\tilde{M}}_{\mathsf{disc}}(\pi)$ for each $\pi \in \Pi_{\mathsf{disc},-}(\tilde{M}^1)$;
- for $\lambda \in i(\mathfrak{a}_M^G)^*$, the global weighted character

$$J_{\tilde{M}}(\pi_\lambda, f) = \mathrm{tr}\left(\mathcal{J}_M(\pi_\lambda, \tilde{P}) I_{\tilde{P}}^{\tilde{G}}(\pi_\lambda, f) \right),$$

  where

  - $\pi_\lambda := \pi \otimes \exp^{\langle \lambda, H_{\tilde{M}}(\cdot) \rangle}$,
  - P is a parabolic with Levi component M,
  - $\mathcal{J}_M(\pi_\lambda, \tilde{P})$ is an intertwining operator coming from $\mathcal{M}_M(\tilde{P}, \lambda)$.
  - $I_{\tilde{P}}^{\tilde{G}}(-)$ is the normalized parabolic induction.

Warning: $\Pi_{\mathsf{disc},-}(\tilde{M}^1)$ is not necessarily contained in the discrete spectrum.

Theorem

We have

$$J(f) = \sum_{M \in \mathcal{L}(M_0)} \frac{|W_0^M|}{|W_0^G|} \sum_{\pi \in \Pi_{\mathsf{disc},-}(\tilde{M}^1)} \int_{i(\mathfrak{a}_M^G)^*} a_{\mathsf{disc}}^{\tilde{M}}(\pi_\lambda) J_{\tilde{M}}(\pi_\lambda, f) d\lambda.$$

Remark

To make this integral absolutely convergent, Arthur put some restriction on the infinitesimal characters of the ∞-part of π. This seems to be avoidable by the work of Finis-Lapid-Müller.

Local unitary weighted characters

Let

- V: a finite set of places such that
  - either V contains some $v|\infty$, or
  - the places in V have the same residual characteristic > 0;
- $P = MU$: a parabolic subgroup with Levi component $M \supset M_0$.

For $\pi \in \Pi(\tilde{M}_V)$ unitary and genuine, there is an operator $\mathcal{M}_M(\pi, \tilde{P})$ acting on the space of $I_{\tilde{M}}^{\tilde{G}}(\pi)$ such that

$$J_{\tilde{M}_V}(\pi, f_V) := \mathrm{tr}\left(\mathcal{M}_M(\pi, \tilde{P}) I_{\tilde{M}}^{\tilde{G}}(\pi, f_V)\right)$$

is well-defined. It does not depend on P.

Remark: $\mathcal{M}_M(\pi, \tilde{P})$ is defined in terms of local intertwining operators and Harish-Chandra's μ-functions; they are canonical objects attached to (M, π).

When $M = G$, we get the usual characters.

Compression of coefficients

Goal: re-index the fine spectral expansion by the local objects $\pi \in \Pi(\tilde{M}_V)$.

Theorem

For any Levi $L \supset M_0$, one can define

- a space $\Pi_-(\tilde{L}^1, V)$ of genuine representations of $\tilde{L}_V$, endowed with a measure,
- coefficients $a^{\tilde{L}}(\pi)$ for $\pi \in \Pi_-(\tilde{L}^1, V)$,

such that for $f = f_V f_{K^V}$ where f_{K^V} is the unit of the antigenuine spherical Hecke algebra w.r.t. $(\tilde{G}^V, K^V)$, we have

$$J(f) = \sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \int_{\Pi_-(\tilde{L}^1, V)} a^{\tilde{L}}(\pi) J_{\tilde{L}}(\pi, 0, f_V).$$

Here $J_{\tilde{L}}(\pi, 0, f_V)$ is the "Fourier coefficient at 0" of $J_{\tilde{L}}(\pi_\lambda, f_V)$.

The compressed coefficients $a^{\tilde{L}}(\pi)$ is defined using

1. the global discrete coefficients $a^{\tilde{M}}_{\text{disc}}(\sigma)$, for various $M \subset L$ and $\sigma \in \Pi_{\text{disc},-}(\tilde{M}^1)$;
2. the normalizing factors $r^L_M(c)$ (for the intertwining operators) of various genuine unramified representations c of $\tilde{G}^V$ w.r.t. K^V.

In view of the works of Langlands and Shahidi, $r^G_M(c)$ is expected to be related to L-functions of some linear group. Cf. [McNamara].

Remark

Note the formal resemblance between the refined geometric and spectral expansions with compressed coefficients.

Invariant trace formula for covers

Let V be a set of places containing the archimedean and ramified ones.

Motivation

In harmonic analysis, we usually choose test functions f_V that are defined only through their

- orbital integrals $J_{\tilde{G}_V}(\tilde{\gamma}, f_V)$, or
- characters $J_{\tilde{G}_V}(\pi, f_V)$ (eg. the Euler-Poincaré functions).

These distributions $f_V \mapsto J_{\tilde{G}_V}(\pi, f_V)$ (resp. $J_{\tilde{G}_V}(\tilde{\gamma}, f_V)$) are weakly dense in the space of invariant distributions on $\tilde{G}_V$.

Goal

Express the distributions $J_{\tilde{L}}(\cdots)$ in the refined trace formula in terms of invariant distributions on Levi subgroups.

Arthur's idea

Let J be the distribution of the refined trace formula, viewed as a distribution on $\tilde{G}_V$ by

$$C_c^\infty(\tilde{G}_V) \ni f_V \longmapsto f := f_V f_{K^V} \in C_c^\infty(\tilde{G}).$$

1. For each Levi L, choose a suitable space of test functions $\mathcal{H}_{\tilde{L}}$ together with a surjection $\mathcal{H}_{\tilde{L}} \to I\mathcal{H}_{\tilde{L}}$.
2. Find a good linear map $\phi_{\tilde{L}} : \mathcal{H}_{\tilde{G}} \to I\mathcal{H}_{\tilde{L}}$, satisfying similar identities under conjugation as the weighted characters/orbital integrals.
3. By induction, define a distribution $I = I^{\tilde{G}}$ so that
   - I is invariant (follows from the conditions above on $\phi_{\tilde{L}}$);
   - I factors through $I\mathcal{H}_{\tilde{G}}$;
   - we have the decomposition

$$J(f) = \sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} I^{\tilde{L}}(\phi_{\tilde{L}}(f_V)), \quad f \in \mathcal{H}_{\tilde{G}},$$

We shall apply a similar procedure to the distributions $J_{\tilde{M}}(\cdots)$ in the refined expansions, using Levi subgroups $L \supset M$. This will yield invariant distributions $I_{\tilde{L}_V}(\tilde{\gamma}, \cdot)$, $I_{\tilde{L}_V}(\pi, \cdot)$.

Invariant trace formula

$$\sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \sum_{\gamma \in \Gamma(\tilde{L}^1, V)} a^{\tilde{L}}(\tilde{\gamma}) I_{\tilde{L}}(\tilde{\gamma}, f_V) = I(f)$$
$$= \sum_{L \in \mathcal{L}(M_0)} \frac{|W_0^L|}{|W_0^G|} \int_{\Pi_-(\tilde{L}^1, V)} a^{\tilde{L}}(\pi) I_{\tilde{L}}(\pi, 0, f_V).$$

This is almost a formal consequence of the recipe above.

Remark

For $L = G$, the distributions $I_{\tilde{G}}(\cdots) = J_{\tilde{G}}(\cdots)$ are the usual characters and orbital integrals.

Choice of $\phi_{\tilde{L}} : \mathcal{H}_{\tilde{L}} \to I\mathcal{H}_{\tilde{L}}$

Following Arthur, we take

- $\mathcal{H}_{\tilde{L}}$ to be the space of $\tilde{K}_V$-finite (left and right), C_c^∞ antigenuine functions on $\tilde{L}_V^1$;
- $I\mathcal{H}_{\tilde{L}}$ to be a space of $\mathbb{C}$-valued functions on the tempered genuine dual of $\tilde{L}_V^1$, characterized by trace Paley-Wiener theorems, so that for any $f_V \in \mathcal{H}_{\tilde{L}}$,
$$\pi \mapsto \mathrm{trace}(\pi(f_V))$$
lies in $I\mathcal{H}_{\tilde{L}}$;
- $\phi_{\tilde{L}}$ to be the map sending f_V to
$$\pi \mapsto J_{\tilde{L}}(\pi, 0, f_V).$$

Recall: $J_{\tilde{L}}(\pi, 0, f_V)$ is the 0-th Fourier coefficient of the weighted character $\lambda \mapsto J_{\tilde{L}}(\pi_\lambda, f_V)$.

1. To show $\phi_{\tilde{L}}$ has image in $I\mathcal{H}_{\tilde{L}}$, a detailed analysis of weighted characters is needed.
2. We also need to assume the trace Paley-Wiener theorem of $\tilde{K}_V$-finite functions.
3. To show that the invariant distribution

$$I(f_V) := J(f) - \sum_{L \neq G} \frac{|W_0^L|}{|W_0^G|} I^{\tilde{L}}(\phi_{\tilde{L}}(f_V))$$

   factors through $I\mathcal{H}_{\tilde{G}}$, Arthur uses a global argument à la Kazhdan, via the trace formula. For some technical reason, this does not work for covers except for $G = \mathrm{GL}(n)$ or $\tilde{G} = \widetilde{\mathrm{Sp}}(2n)$ (the twofold cover of $\mathrm{Sp}(2n)$).
4. We use a purely local argument via the invariant local trace formula for covering groups.

Trace Paley-Wiener theorem

Assumption

For all Levi $L \subset G$, the trace Paley-Wiener theorem characterizing the image of

$$f_V \longmapsto [\pi \mapsto \text{trace}(\pi(f_V))]$$

for $f_V \in \mathcal{H}_{\tilde{L}}$ holds, where π is a genuine tempered irrep of $\tilde{L}^1_V$.

1. This can be reduced to the case $V = \{v\}$, for genuine tempered irreps of $\tilde{L}_v$.
2. When v is nonarchimedean: simply copy the arguments of Bernstein-Deligne-Kazhdan.
3. When v is archimedean: some arguments of Clozel-Delorme seem problematic for covers!

For archimedean v, some special cases can be check by *ad hoc* arguments:

- $G = \mathsf{GL}(n)$ (used in Mezo's work on metaplectic correspondence),
- $G = \mathsf{SL}(2)$ (Hiraga-Ikeda),
- $\tilde{G} = \widetilde{\mathsf{Sp}}(2n)$ the twofold cover of $\mathsf{Sp}(2n)$ (Weil...),
- G unitary group.

Cuspidal test functions

Let v be a place of F, $f_v \in C_c^\infty(\tilde{G}_v)$ be antigenuine.

Definition

We say f_v is cuspidal if trace$(\pi(f)) = 0$ for any π of the form

$$\pi = I_{\tilde{P}}^{\tilde{G}}(\sigma)$$

where $P \neq G$ and σ is a tempered genuine irrep of $\tilde{M}_v$.

Let $f_V = \prod_{v \in V} f_v \in \mathcal{H}_{\tilde{G}}$, we say f_V is cuspidal at v if f_v is cuspidal.
Recall: the invariant distribution $f_V \mapsto I(f_V)$ has two expansions: spectral and geometric.

Theorem

Let $f_V = \prod_{v \in V} f_v \in \mathcal{H}_{\tilde{G}}$. Put $f := f_V f_{K^V} \in C_c^\infty(\tilde{G}^1)$

1. If f_V is cuspidal at one place, then the spectral expansion of $I(f_V)$ becomes
$$\sum_{\pi_V \in \Pi_{\text{disc},-}(\tilde{G}^1, V)} a^{\tilde{G}}(\pi_V) J_{\tilde{G}}(\pi_V, f_V)$$
or in global terms:
$$\sum_{\pi \in \Pi_{\text{disc},-}(\tilde{G}^1)} a^{\tilde{G}}_{\text{disc}}(\pi) J_{\tilde{G}}(\pi, f).$$
2. If f is cuspidal at two places, then the geometric expansion of $I(f)$ becomes
$$\sum_{\tilde{\gamma} \in \Gamma(\tilde{G}^1, V)} a^{\tilde{G}}(\tilde{\gamma}) J_{\tilde{G}}(\tilde{\gamma}, f).$$