Local Gross-Prasad conjecture over archimedean local fields

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Setting of the Conjecture

- F a local field, $\mathrm{char}(F) = 0$;
- S is an even-dimensional and split quadratic space$/F$;
- D is an anisotropic line$/F$;
- V, W non-deg quadratic spaces $/F$ s.t. $V = W \oplus^{\perp} S \oplus^{\perp} D$.

Gross-Prasad triple (G, H, ξ)

- $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, $H = \Delta\mathrm{SO}(W) \ltimes N$;
- ξ a unitary character on $H(F)$ induced from a generic unitary character ξ_N on $N(F)$.

Example

- **Codimension-one case**: $S = 0$. *In this case,* $\dim V = \dim W + 1$, $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, $H = \Delta\mathrm{SO}(W)$ *and ξ is the trivial character of $H(F)$.*
- **Whittaker Case**: $\dim W = 0, 1$. *In this case, $G = \mathrm{SO}(V)$ is quasi-split, $H = N$ maximal unipotent and ξ is a Whittaker character.*

Multiplicity For an irreducible admissible(nonarchimedean)/ Casselman-Wallach(archimedean) representation π of $G(F)$

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi, \xi)$$

Theorem (AGRS10, Wald11, GGP12; SZ12, JSZ11)

$$m(\pi) \leqslant 1$$

Pure inner forms of spherical pairs

For every $\alpha \in H^1(F, H) \to H^1(F, G)$, we have pure inner forms

- $G_\alpha = \mathrm{SO}(V_\alpha) \times \mathrm{SO}(W_\alpha)$;
- $H_\alpha = \Delta\mathrm{SO}(W_\alpha) \ltimes N$;

where $V_\alpha = W_\alpha \oplus^\perp H \oplus^\perp D$. Together with ξ_α induced by ξ_N, we obtain a Gross-Prasad triple

$$(G_\alpha, H_\alpha, \xi_\alpha)$$

Vogan packet

Given a generic L-parameter $\varphi : \mathcal{W}_F \to {}^LG(= {}^LG_\alpha)$, the LLC gives L-packets $\Pi_\varphi(G_\alpha)$ for every $\alpha \in H^1(F, G)$

$$\Pi_\varphi^{\mathrm{Vogan}} = \coprod_{\alpha \in H^1(F,G)} \Pi_\varphi(G_\alpha)$$

It was proved by Shelstad over archimedean fields and conjectured by Vogan over nonarchimedean fields that, fixing a Whittaker datum, there exists a non-deg $\mathbb{Z}/2\mathbb{Z}$-bilinear pairing

$$\Pi_\varphi^{\mathrm{Vogan}} \times \mathcal{S}_\varphi \to \{\pm 1\}$$

where

$$\mathcal{S}_\varphi = \pi_0(\mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\varphi)))$$

This gives

$$\Pi_\varphi^{\mathrm{Vogan}} \longleftrightarrow \text{characters of } \mathcal{S}_\varphi$$

Local Gross-Prasad conjecture

Conjecture (GP92,GP94)

Given a generic parameter φ

- **Multiplicity-one for Vogan packets** *There exists exactly one representation in the Vogan packet* $\Pi_\varphi^{\mathrm{Vogan}}$ *with multiplicity equal to one.*
- **Epsilon-Dichotomy** *Gross and Prasad defined a character* $\chi_{\varphi,H}$ *of* $\mathcal{S}_\varphi$ *using local epsilon factors. They also specified a choice of Whittaker datum. Under this Whittaker datum, the distinguished representation in the Vogan packet corresponds to* $\chi_{\varphi,H}$.

Example: $\mathrm{SO}(3) \times \mathrm{SO}(2)$, $F = \mathbb{R}$, discrete series case

- $G = \mathrm{SO}(3,0) \times \mathrm{SO}(2,0)$, $H = \Delta\mathrm{SO}(2,0)$, $\xi = 1_{H(\mathbb{R})}$
- $G_\alpha = \mathrm{SO}(3,0) \times \mathrm{SO}(2,0)$ or $\mathrm{SO}(1,2) \times \mathrm{SO}(0,2)$
- ${}^L G_\alpha = \mathrm{Sp}(2,\mathbb{C}) \times \mathrm{O}(2,\mathbb{C}) \subset \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$
- For non-negative integer l, $\varphi_l : \mathcal{W}_\mathbb{R} = \mathbb{C}^\times \coprod \mathbb{C}^\times j \to \mathrm{GL}(2,\mathbb{C})$

$$z \mapsto \begin{pmatrix} |z|^{2t}(\frac{z}{|z|})^l & \\ & |z|^{2t}(\frac{z}{|z|})^{-l} \end{pmatrix} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ (-1)^l & 0 \end{pmatrix}$$

- When l is even, $\varphi_l : \mathcal{W}_\mathbb{R} \to \mathrm{O}(2,\mathbb{C})$, LLC gives

$$\text{One-dim } \mathrm{SO}(2,0)(\mathbb{R})\text{-repn } \pi_l : \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \mapsto e^{i\theta l\pi}$$

- When l is odd, $\varphi_l : \mathcal{W}_\mathbb{R} \to \mathrm{Sp}(2,\mathbb{C})$, LLC gives

$$l\text{-dim } \mathrm{SO}(3,0)(\mathbb{R})\text{-repn } F_l \text{ with highest weight } \frac{l-1}{2}$$

$$\text{A discrete series repn } D_l \text{ of } \mathrm{SO}(1,2)(\mathbb{R}) = \mathrm{PGL}(2,\mathbb{R})$$

Example: $\mathrm{SO}(3) \times \mathrm{SO}(2)$, $F = \mathbb{R}$, discrete series case

Given an odd integer $l \geqslant 1$ and an even integer $n \geqslant 0$

$$m(F_l \otimes \pi_n) = \mathrm{Hom}_{H(F)}(F_l \otimes \pi_n, 1) = 1 \text{ iff } l > n$$

$$m(D_l \otimes \pi_n) = \mathrm{Hom}_{H(F)}(D_l \otimes \pi_n, 1) = 1 \text{ iff } l < n$$

The component group $\mathcal{S}_{\varphi_l} = \{\pm 1\}$

$$\begin{aligned} \chi_{\varphi_l \times \varphi_n, H}(-1, 1) &= (-1)^{\frac{4}{4}} \varepsilon(\varphi_l \otimes \varphi_n) = -\varepsilon(\varphi_{l+n} \oplus \varphi_{|l-n|}) \\ &= -i^{l+n+1} i^{|l-n|+1} = i^{2max\{l,n\}} \\ &= \begin{cases} 1 & \text{if } l < n \\ -1 & \text{if } l > n \end{cases} \end{aligned}$$

Related results (local Gross-Prasad conjecture)

Over nonarchimedean local fields

- Waldspurger proved the **full conjecture** for **tempered parameters**
- Mœglin and Waldspurger proved the **full conjecture** for **generic parameters**

Over archimedean local fields

- Möllers proved the **full conjecture** for **codimension-one case** over $\mathbb{C}$
- Luo proved the **multiplicity-one part** of the conjecture for **tempered parameters** over $\mathbb{R}$
- C.-Luo proved the **epsilon-dichotomy part** of the conjecture for **tempered parameters** over $\mathbb{R}$
- C. proved the **full conjecture** for **generic parameters** over $\mathbb{R}$ and $\mathbb{C}$

Related results (local Gan-Gross-Prasad conjecture for unitary groups)

Over nonarchimedean local fields

- Beuzart-Plessis proved the **full conjecture** for **tempered parameters**
- Gan and Ichino proved the **full conjecture** for **generic parameters**

Over archimedean local fields

- Beuzart-Plessis proved the **multiplicity-one part** of the conjecture for **tempered parameters**
- Xue proved the **full conjecture** for **tempered parameters** using theta correspondence
- Xue proved the **full conjecture** for **generic parameters**

(These results are before our proof of Gross-Prasad conjecture.)

Waldspurger's proof for Epsilon-dichotomy (tempered, nonarchimedean)

Step 1: Local trace formula (on $\mathrm{SO}(V) \times \mathrm{SO}(W)$)

$$J_{\mathrm{spec}}(f) = J_{\mathrm{geom}}(f) \Longrightarrow m(\pi) = m_{\mathrm{geom}}(\pi)$$

Step 2: Express the epsilon factor in terms of the Harish-Chandra characters $\Theta_{\Pi_\varphi(G)}$ using twisted endoscopy and twisted local trace formula (on $\mathrm{GL}(n) \rtimes \theta$, $\theta =$transpose inverse)

$$J_{\mathrm{spec}}(\widetilde{f}) = J_{\mathrm{geom}}(\widetilde{f}) \Longrightarrow m(\widetilde{\pi}) = m_{\mathrm{geom}}(\widetilde{\pi})$$

Step 3: Study $m_{\mathrm{geom}}(\pi)$ under endoscopy with a multiplicity formula.

Our proof (tempered, $F = \mathbb{R}$)

Step 1: Local trace formula (proved by Luo)

Step 2: (Reductions) Easier over archimedean fields. When $\dim V > 3$, the parameter φ_V is either be **parabolic type** or **endoscopic type**. In both case, we are able to reduce the question to smaller cases. So the question can be reduced to the Waldspurger's model.

Step 3: (For reduction of endoscopic type) Instead of considering the geometric multiplicity for one pure inner form, we sum over the geometric multiplicity of all pure inner forms with the same Kottwitz sign.

Classification of the parameter φ_V ($F = \mathbb{R}$)

When dim $V > 3$ the parameter φ_V of $\mathrm{SO}(V)$ can be classified as the following:

- Parabolic Type: $\varphi_V = \varphi_V^{\mathrm{GL}} \oplus \varphi_{V_0} \oplus (\varphi_V^{\mathrm{GL}})^\vee$, $\varphi_V^{\mathrm{GL}} \neq 0$. In this case, let $\Pi_{\varphi_V^{\mathrm{GL}}} = \{\sigma\}$, the map
$$\pi_0 \mapsto \sigma \times \pi_0 = I_P^G(\sigma \otimes \pi_0)$$
defines an isomorphism
$$\Pi_{\varphi_{V_0}}^{\mathrm{Vogan}} \to \Pi_{\varphi_V}^{\mathrm{Vogan}}$$
- Endoscopic Type: Exists $s \in \mathcal{S}_{\varphi_V}$ such that the endoscopic group $G'_V = \mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$ of $G_V = \mathrm{SO}(V)$ determined by s is smaller than $\mathrm{SO}(V)$, that is,
$${}^L G'_V \subsetneq {}^L G_V \quad \mathrm{Cent}_{\widehat{G_V}}(s)^o = \widehat{G'_V}$$

Geometric multiplicity (Gross-Prasad case)

For a quasi-character Θ of $G(\mathbb{R})$, we define

$$m_{\mathrm{geom}}(\Theta) = \int_{\Gamma(G,H)} D^G(x)^{1/2} c_\Theta(x) \Delta(x)^{-1/2} dx$$

- $\Gamma(G,H)$ the space of semisimple $G(\mathbb{R})$-conjugacy classes that intersect $H(\mathbb{R})$;
- $D^G(x) = |\det(1 - \mathrm{Ad}(x))|_{\mathfrak{g}/\mathfrak{g}_x}|$
  $D^H(x) = |\det(1 - \mathrm{Ad}(x))|_{\mathfrak{h}/\mathfrak{h}_x}|$
- $\Delta(x) = D^G(x) D^H(x)^{-2}$
- $c_\Theta(x) = c_{\Theta,\mathcal{O}}(x)$ the germ (coefficient in character expansion) with respect to $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)$ which is uniformly selected.

In particular, for tempered representation π, we let

$$m_{\mathrm{geom}}(\pi) = m_{\mathrm{geom}}(\Theta_\pi)$$

Relation to Spectral Multiplicity

Theorem (L 2021)

For every tempered representation $\pi = \pi_V \widehat{\otimes} \pi_W$ *of* $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$

$$m(\pi) = m_{\mathrm{geom}}(\pi)$$

Reduction: parabolic situation

Theorem (L 2021)

Let $V = X \oplus V_0 \oplus X^\vee$*, for every tempered representation* π_{V_0} *of* $\mathrm{SO}(V_0)$*,* π_W *of* $\mathrm{SO}(W)$ *and* σ *of* $\mathrm{GL}(X)$

$$m_{\mathrm{geom}}(\pi_{V_0}\widehat{\otimes}\pi_W) = m_{\mathrm{geom}}((\sigma \times \pi_{V_0})\widehat{\otimes}\pi_W)$$

Therefore

$$m(\pi_{V_0}\widehat{\otimes}\pi_W) = m((\sigma \times \pi_{V_0})\widehat{\otimes}\pi_W)$$

Proposition (Reductions for parabolic types)

Suppose $\varphi_V = \varphi_V^{\mathrm{GL}} \oplus \varphi_{V_0} \oplus (\varphi_V^{\mathrm{GL}})^\vee$ *is of parabolic type and the Conjecture holds for* $\varphi_{V_0} \times \varphi_W$*, then the Conjecture holds for* $\varphi_V \times \varphi_W$*.*

Virtual tempered representations

A **virtual tempered representation** (VTR) Σ of $G(\mathbb{R})$ is a finite $\mathbb{C}$-linear combination $\sum_i a_i[\pi_i]$ of infinitesimal equivalence classes of irreducible tempered representations π_i.

A VTR $\Sigma = \sum_i a_i[\pi_i]$ is uniquely determined by the quasi-character $\Theta_\Sigma = \sum_i a_i \Theta_{\pi_i}$.

We can define the multiplicities for VTR as

$$m(\Sigma) = \sum_i a_i m(\pi_i) \quad m_{\mathrm{geom}}(\Sigma) = \sum_i a_i m_{\mathrm{geom}}(\pi_i) = m_{\mathrm{geom}}(\Theta_\Sigma)$$

Stable geometric multiplicity

We define the **stable geometric multiplicity**

$$m^S_{\mathrm{geom}}(\Theta) = \int_{\Gamma(G,H)} D^G(x)^{1/2} c^{\mathrm{Stab}}_{\Theta}(x)\Delta(x)^{-1/2}dx$$

Here

$$c^{\mathrm{Stab}}_{\Theta}(x) = \frac{1}{|\mathrm{Nil}(\mathfrak{g}_x)|}\sum_{\mathcal{O}\in\mathrm{Nil}(\mathfrak{g}_x)} c_{\Theta,\mathcal{O}}(x)$$

Endoscopic transfer

Let $G'(\mathbb{R})$ be an endoscopic group of $G(\mathbb{R})$. A character Θ^G is called the **endoscopic transfer** of $\Theta^{G'}$ if we have

$$\Theta^G(x) = \sum_y \Delta(y,x)\Theta^{G'}(y) \quad x \in G(\mathbb{R})/\mathrm{conj}, y \in G'(\mathbb{R})/\mathrm{stconj}$$

summation over stable $G'(\mathbb{R})$-conjugacy classes. Here Δ is the Langlands-Shelstad transfer factor.

Theorem (Shelstad)

Let $\varphi_G : \mathcal{W}_\mathbb{R} \to {}^L G$ be a generic L-parameter that induces $\varphi_{G'} : \mathcal{W}_\mathbb{R} \to {}^L G'$. Moreover, $\mathrm{Cent}_{\widehat{G}}(s)^o = \widehat{G'}$. The twisted character

$$e(G)\Theta^\chi_{\varphi_G} = e(G) \sum_{\pi \in \Pi_{\varphi_G}(G)} \chi_\pi(s)\Theta_\pi$$

on $G(\mathbb{R})$ is the endoscopic transfer of the stable character $\Theta^1_{\varphi_{G'}} = \sum_{\pi \in \Pi_{\varphi_{G'}}(G')} \Theta_\pi$ of $G'(\mathbb{R})$.

Key Multiplicity formula

Theorem (CL, 2022)

Given VTRs $\Sigma_{V_\alpha}, \Sigma_{W_\alpha}, \Sigma_{V_\pm}, \Sigma_{W_\pm}$*, suppose*

1. $\Theta_{\Sigma_{V_+}}, \Theta_{\Sigma_{V_-}}, \Theta_{\Sigma_{W_+}}, \Theta_{\Sigma_{W_-}}$ *are stable characters of* $\mathrm{SO}(V_+)$, $\mathrm{SO}(V_-)$, $\mathrm{SO}(W_+)$, $\mathrm{SO}(W_-)$ *respectively;*
2. *For every* $\alpha \in H^1(F, H)$, $e(G_{V_\alpha})\Theta_{\Sigma_{V_\alpha}}$ *are the endoscopic transfer of* $\Theta_{\Sigma_{V_+}} \times \Theta_{\Sigma_{V_-}}$ *and* $e(G_{W_\alpha})\Theta_{\Sigma_{W_\alpha}}$ *are the endoscopic transfer of* $\Theta_{\Sigma_{W_+}} \times \Theta_{\Sigma_{W_-}}$*;*

Then for $e_0 = \pm 1$*, we have*

$$\sum_{\substack{\alpha \in H^1(\mathbb{R}, H) \\ e(G_\alpha) = e_0}} m_{\mathrm{geom}}(\Sigma_{V_\alpha}, \Sigma_{W_\alpha})$$

$$= \frac{1}{2}(e_0 \cdot m^S_{\mathrm{geom}}(\Sigma_{V_+}, \Sigma_{W_+}) m^S_{\mathrm{geom}}(\Sigma_{V_-}, \Sigma_{W_-}) + m^S_{\mathrm{geom}}(\Sigma_{V_-}, \Sigma_{W_+}) m^S_{\mathrm{geom}}(\Sigma_{V_+}, \Sigma_{W_-}))$$

Choice of Σ's

Given a parameter φ_V of $\mathrm{SO}(V)$ and $s \in \mathcal{S}_{\varphi_V}$, we let

$$\Sigma^s_{V_\alpha} = \sum_{\pi \in \Pi_\varphi(\mathrm{SO}(V))} \chi_\pi(s)\pi$$

for $\alpha \in H^1(\mathbb{R}, \mathrm{SO}(V))$.

Let $\mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$ be the endoscopic group of $\mathrm{SO}(V)$ determined by the eigenspace decomposition of s. We take

$$\Sigma^1_{V_+} = \sum_{\pi \in \Pi_{\varphi_+}(\mathrm{SO}(V_+))} \pi, \quad \Sigma^1_{V_-} = \sum_{\pi \in \Pi_{\varphi_-}(\mathrm{SO}(V_-))} \pi$$

The VTRs $\Sigma^s_{V_\alpha}, \Sigma^s_{W_\alpha}, \Sigma^1_{V_\pm}, \Sigma^1_{W_\pm}$ satisfy the assumptions (1)(2).

Reduction: endoscopic type

We define the stable multiplicity

$$m^S_{V,W} = \sum_{\alpha \in H^1(\mathbb{R},H)} m(\Sigma^1_{V_\alpha}, \Sigma^{-1}_{W_\alpha})$$

Corollary

For quasi-split $\mathrm{SO}(V)$ *and* $\mathrm{SO}(W)$

$$m^S_{V,W} = m^S_{\mathrm{geom}}(\Sigma^1_V, \Sigma^1_W)$$

Corollary

$$\sum_{\alpha \in H^1(\mathbb{R},H)} m(\Sigma^{s_V}_{V_\alpha}, \Sigma^{s_W}_{W_\alpha}) = m^S_{V_-,W_+} m^S_{V_+,W_-}$$

Reduction: endoscopic situation

Suppose the conjecture holds for all possible (V_-, W_+) and (V_+, W_-), we have

$$\sum_{\pi \in \Pi^{\mathrm{Vogan}}(G)} \chi_\pi(s) m(\pi) = \sum_{\alpha \in H^1(\mathbb{R}, H)} m(\Sigma^{s_V}_{V_\alpha}, \Sigma^{s_W}_{W_\alpha})$$
$$= m^S_{V_-, W_+} m^S_{V_+, W_-} = \chi_{\varphi, H}(s)$$

Both sides are characters and the equality holds for all s such that the endoscopic group $\mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$ determined by s is not equal to $\mathrm{SO}(V)$, equivalently

$$s_V \in \mathcal{S}_{\varphi_V} - \mathbb{C} \cdot 1_\varphi \cap \mathcal{S}_{\varphi_V}$$

So the equality holds for all $s = s_V \times s_W \in \mathcal{S}_\varphi$.

Proposition (Reductions for endoscopic type)

Suppose φ_V is of endoscopic type and the Conjecture holds for $\varphi_{V'} \times \varphi_{W'}$ such that $\dim V' < \dim V$. *Then the Conjecture holds for $\varphi_V \times \varphi_W$.*

Proof for the key multiplicity formula

This idea follows from Waldspurger's work

- **Step 1**: Prove a germ formula

$$c_{\Theta}(x) = \lim_{t\to 0^+}\Big[\frac{D^{G_x}(tX_{\mathrm{qd},x})}{|W_{T_{\mathrm{qd},x}}|}\Theta(x\exp(tX_{\mathrm{qd},x}))$$

$$+\frac{D^{G_x}(tX_x)}{2|W_{T_x}|}\eta(\Theta(x\exp(tX_x^+)) - \Theta(x\exp(tX_x^-)))\Big]$$

  and apply the germ formula in the definition of geometric multiplicity.
  (Waldspurger's proof for the germ formula used some properties of smooth transfer on Lie algebra which he proved with fundamental lemma.)
- **Step 2**: Use endoscopy to write the characters $\Theta(x\exp(tX_{\mathrm{qd},x}))$, $\Theta(x\exp(tX_x^{\pm}))$ in terms of stable characters on the endoscopic group.
- **Step 3**: Rearrange the terms to obtain an expression in terms of stable geometric multiplicity.

Proof for generic case: F nonarchimedean

Mœglin and Waldspurger's framework

Step 1: A structure theorem showing every representation in generic packets can be expressed as a parabolic induction;
(The proof for the structure theorem uses **Casselman-Shahidi's standard module conjecture** (proved by Muić) and an **irreducibility criterion**)

Step 2: Reduction from co-dimension one cases to tempered cases with a **mathematical induction**;
(The induction steps were proved with a **multiplicity formula**.)

Step 3: Reduction from general cases to co-dimension one cases using the **multiplicity formula** in **Step 2**.

Proof for generic case: $F = \mathbb{R}$

Step 1: **standard module conjecture** was proved by Vogan; **irreducibility criterion** was proved by Speh and Vogan.

Step 2: With **a multiplicity formula** as in Mœglin and Waldspurger ($s_{\pi_W,\sigma} = s_{\pi_W} + s_\sigma$), the **mathematical induction** won't work.
I proved **a refined multiplicity formula** ($s_{\pi_W,\sigma} = s_{\pi_W} - s_\sigma$) so that the mathematical induction works.
(Multiplicity formula are proved using Schwartz homologies)

Step 3: Reduction from general cases to co-dimension one cases using a **multiplicity formula**.

The **multiplicity formula** are in the form of

$$m((|\det|^s\sigma \times \pi_V)\widehat{\otimes}\pi_W) = m(\pi_V\widehat{\otimes}\pi_W) \quad \text{for } \mathrm{Re}(s) \geqslant s_{\pi_W,\sigma}$$

Proof for generic case: $F = \mathbb{C}$

(Codimension-one case proved by Möllers.) Notice that $|\Pi_\varphi^{\mathrm{Vogan}}| = 1$.

- **Step 1** We can find a Borel subgroup $B = B_V \times B_W$ of G such that $H \cap B = 1$ and BH is Zariski-open in G. So we can a define $(B(F) \times H(F), \delta_{B(F)}^{1/2}\sigma \times \xi)$-equivalent tempered measure
$$\mu = \delta_{B(F)}^{-1/2}\sigma^{-1}(b)\xi(h)dbdh.$$
can be constructed on $B(F)H(F)$
- **Step 2** From [GSS 16], this measure can be "extended" to a nonzero $(B(F) \times H(F), \delta_{B(F)}^{1/2}\sigma \times \xi)$-equivalent tempered distribution on $G(F)$.
- **Step 3** We can construct a nonzero element in
$$\mathrm{Hom}_H(I_B^G(\sigma), \xi)$$
with this distribution.

Thank you!