Local Models For Moduli Of Global and Local G-Shtukas

Esmail Arasteh Rad
Univ. Münster

July 22, 2020

Outline Of The Talk

Outline Of The Talk

- Beginning
- Middle
- End

§Shimura data and $\nabla\mathcal{H}$-data

§Shimura data and $\nabla\mathcal{H}$-data

- A Shimura datum $(\mathbb{G}, X, K)$
  - a reductive group $\mathbb{G}$ over $\mathbb{Q}$ with center Z,
  - $\mathbb{G}(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to \mathbb{G}_{\mathbb{R}}$ for the Deligne torus $\mathbb{S}$,
  - A compact open sub-group $K \subseteq \mathbb{G}(\mathbb{A}_f)$,

§Shimura data and $\nabla\mathcal{H}$-data

- A Shimura datum $(\mathbb{G}, X, K)$
  - a reductive group $\mathbb{G}$ over $\mathbb{Q}$ with center Z,
  - $\mathbb{G}(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to \mathbb{G}_{\mathbb{R}}$ for the Deligne torus $\mathbb{S}$,
  - A compact open sub-group $K \subseteq \mathbb{G}(\mathbb{A}_f)$,
  subject to certain conditions...

§Shimura data and $\nabla\mathcal{H}$-data

- A Shimura datum $(\mathbb{G}, X, K)$
  - a reductive group $\mathbb{G}$ over $\mathbb{Q}$ with center Z,
  - $\mathbb{G}(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to \mathbb{G}_{\mathbb{R}}$ for the Deligne torus $\mathbb{S}$,
  - A compact open sub-group $K \subseteq \mathbb{G}(\mathbb{A}_f)$,

  subject to certain conditions...

$$(\mathbb{G}, X, K) \rightsquigarrow Sh_K(\mathbb{G}, X) = \mathbb{G}(\mathbb{Q})\backslash X \times \mathbb{G}(\mathbb{A}_f)/K$$

- Let's wish a modular interpretation exists...

- Let's wish a modular interpretation exists...

  "In order to be able to realize all but a handful of Shimura varieties as moduli varieties, we shall need to replace algebraic varieties and algebraic classes by more general objects, namely, by motives" (Milne 198?)

- Let's wish a modular interpretation exists...

  "In order to be able to realize all but a handful of Shimura varieties as moduli varieties, we shall need to replace algebraic varieties and algebraic classes by more general objects, namely, by motives" (Milne 198?)

  inspired by a former observation of P. Deligne

  "Pour interpréter des structures de Hodge de type plus compliqué, on aimerait remplacer les variétés abéliennes par des "motifs" convenables, mais il ne s'agit encore que d'un rêve." (Deligne 1979, p. 248)

§The Journey to the Dreamland of FF

$\mathbb{Q} \leftrightsquigarrow$ smooth proj. curve $C/\mathbb{F}_q$

$$\mathbb{Q} \leftrightsquigarrow \text{smooth proj. curve } C/\mathbb{F}_q$$

There are certain candidates for the true category of motives over function fields.

$$\mathbb{Q} \leftrightsquigarrow \text{smooth proj. curve } C/\mathbb{F}_q$$

There are certain candidates for the true category of motives over function fields.

- Category of t-motives (Anderson 1986)
  Def: $A := \mathbb{F}_q[t]$, L an A-field via $A \to L,\ t \mapsto \theta$ (char. morphism). An effective t-motive of rk r over L is a pair $\underline{M} = (M, \tau)$
  - a free and f.g. A_L-module M of rank r, and
  - $\tau : \sigma^* M := M \otimes_{A_L, \sigma^*} A_L \to M$ (s. th. $(t-\theta)^d$ annihilates $coker\ \tau$).
  Here $\sigma^* : A_L \to A_L, a \otimes b \mapsto a \otimes b^q$.

- Taelman's caegory $t\mathcal{M}^\circ$ (Dissertation 200?)

- Taelman's caegory $t\mathcal{M}^{\circ}$ (Dissertation 200?)
  - Hom$(-,-) \otimes Q$ ($Q = Frac(A)$)

- Taelman's caegory $t\mathcal{M}^\circ$ (Dissertation 200?)
  - $\mathrm{Hom}(-,-) \otimes Q$ ($Q = Frac(A)$)
  - formally invert (tensor powers of) the Carlitz motive $\underline{\mathbf{C}}$.

$$\underline{\mathbf{C}} \text{ Carlitz motive} \leftrightsquigarrow M(\mathbb{G}_m)$$

- Taelman's caegory $t\mathcal{M}^{\circ}$ (Dissertation 200?)
  - $\mathrm{Hom}(-,-) \otimes Q$ ($Q = Frac(A)$)
  - formally invert (tensor powers of) the Carlitz motive $\underline{\mathbf{C}}$.

$$\underline{\mathbf{C}} \text{ Carlitz motive} \leftrightsquigarrow M(\mathbb{G}_m)$$

The resulting category $t\mathcal{M}^{\circ}$ together with the obvious fiber functor $\omega : t\mathcal{M}^{\circ} \to Q - \textit{vector spaces}$ provides a tannakian category which is a candidate for the analogous motivic category over function fields. Still one may naturally want:

- multiplication by a Dedekind domain which is strictly bigger than $\mathbb{F}_q[t]$,

- multiplication by a Dedekind domain which is strictly bigger than $\mathbb{F}_q[t]$,
- to construct a category analogous to the category of (mixed) motives over a general base, and

- multiplication by a Dedekind domain which is strictly bigger than $\mathbb{F}_q[t]$,
- to construct a category analogous to the category of (mixed) motives over a general base, and
- to geometrize this category.

- multiplication by a Dedekind domain which is strictly bigger than $\mathbb{F}_q[t]$,
- to construct a category analogous to the category of (mixed) motives over a general base, and
- to geometrize this category.

To handle the above

- multiplication by a Dedekind domain which is strictly bigger than $\mathbb{F}_q[t]$,
- to construct a category analogous to the category of (mixed) motives over a general base, and
- to geometrize this category.

To handle the above

1. replace t-motives with A-motives, make A-motives completed at the place infinity ∞ of a curve C and replace M by vector bundle $\mathcal{M}/C_L$

- multiplication by a Dedekind domain which is strictly bigger than $\mathbb{F}_q[t]$,
- to construct a category analogous to the category of (mixed) motives over a general base, and
- to geometrize this category.

To handle the above

1. replace t-motives with A-motives, make A-motives completed at the place infinity ∞ of a curve C and replace M by vector bundle $\mathcal{M}/C_L$
2. Let's invert further Carlitz(-Hayes) motives (this corresponds to introducing further characteristic sections $s_i : \operatorname{Spec} L \to C$)

- multiplication by a Dedekind domain which is strictly bigger than $\mathbb{F}_q[t]$,
- to construct a category analogous to the category of (mixed) motives over a general base, and
- to geometrize this category.

To handle the above

1. replace t-motives with A-motives, make A-motives completed at the place infinity ∞ of a curve C and replace M by vector bundle $\mathcal{M}/C_L$
2. Let's invert further Carlitz(-Hayes) motives (this corresponds to introducing further characteristic sections $s_i : \operatorname{Spec} L \to C$)

One can then easily see that the resulting category is equivalent with the following category

§Cat Of C-Motives

§Cat Of C-Motives

- Definition
  Let C be a sm. proj. curve over $\mathbb{F}_q$. Fix $\underline{\nu} := (\nu_i) \in C^n$. Let $S \in Sch/\mathbb{F}_q$. A *C-motive* $\underline{\mathcal{M}}$ with char $\underline{\nu}$ over S is a tuple $(\mathcal{M}, \tau_{\mathcal{M}})$

§Cat Of C-Motives

- Definition

  Let C be a sm. proj. curve over $\mathbb{F}_q$. Fix $\underline{\nu} := (\nu_i) \in C^n$. Let $S \in Sch/\mathbb{F}_q$. A *C-motive* $\underline{\mathcal{M}}$ with char $\underline{\nu}$ over S is a tuple $(\mathcal{M}, \tau_{\mathcal{M}})$

    - a loc. free sheaf $\mathcal{M}$ of $\mathcal{O}_{C_S}$-mod of finite rk,

§Cat Of C-Motives

- Definition

  Let C be a sm. proj. curve over $\mathbb{F}_q$. Fix $\underline{\nu} := (\nu_i) \in C^n$. Let $S \in Sch/\mathbb{F}_q$. A *C-motive* $\underline{\mathcal{M}}$ with char $\underline{\nu}$ over S is a tuple $(\mathcal{M}, \tau_{\mathcal{M}})$

  - a loc. free sheaf $\mathcal{M}$ of $\mathcal{O}_{C_S}$-mod of finite rk,
  - $\tau_{\mathcal{M}} : \sigma^* \dot{\mathcal{M}} \tilde{\to} \dot{\mathcal{M}}$ where $\dot{\mathcal{M}}$ is the restriction of $\mathcal{M}$ to $\dot{C}_S$ ($\dot{C} = C \setminus \underline{\nu}$), and $\sigma = id \times \sigma_S$ where $\sigma_S : S \to S$ is the abs. Frob./$\mathbb{F}_q$.

§Cat Of C-Motives

- **Definition**

  Let C be a sm. proj. curve over $\mathbb{F}_q$. Fix $\underline{\nu} := (\nu_i) \in C^n$. Let $S \in Sch/\mathbb{F}_q$. A *C-motive* $\underline{\mathcal{M}}$ with char $\underline{\nu}$ over S is a tuple $(\mathcal{M}, \tau_{\mathcal{M}})$

  - a loc. free sheaf $\mathcal{M}$ of $\mathcal{O}_{C_S}$-mod of finite rk,
  - $\tau_{\mathcal{M}} : \sigma^*\dot{\mathcal{M}} \tilde{\to} \dot{\mathcal{M}}$ where $\dot{\mathcal{M}}$ is the restriction of $\mathcal{M}$ to $\dot{C}_S$ ($\dot{C} = C \setminus \underline{\nu}$), and $\sigma = id \times \sigma_S$ where $\sigma_S : S \to S$ is the abs. Frob.$/\mathbb{F}_q$.

  The set of quasi-morphisms QHom$(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ is given by

$$\begin{array}{ccc} \sigma^*\mathcal{M}_\eta & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M}_\eta \\ \downarrow & & \downarrow \\ \sigma^*\mathcal{N}_\eta & \xrightarrow{\tau_{\mathcal{N}}} & \mathcal{N}_\eta \end{array}$$

§Cat Of C-Motives

- Definition

  Let C be a sm. proj. curve over $\mathbb{F}_q$. Fix $\underline{\nu} := (\nu_i) \in C^n$. Let $S \in Sch/\mathbb{F}_q$. A *C-motive* $\underline{\mathcal{M}}$ with char $\underline{\nu}$ over S is a tuple $(\mathcal{M}, \tau_{\mathcal{M}})$

  - a loc. free sheaf $\mathcal{M}$ of $\mathcal{O}_{C_S}$-mod of finite rk,
  - $\tau_{\mathcal{M}} : \sigma^* \dot{\mathcal{M}} \tilde{\to} \dot{\mathcal{M}}$ where $\dot{\mathcal{M}}$ is the restriction of $\mathcal{M}$ to $\dot{C}_S$ ($\dot{C} = C \setminus \underline{\nu}$), and $\sigma = id \times \sigma_S$ where $\sigma_S : S \to S$ is the abs. Frob.$/\mathbb{F}_q$.

  The set of quasi-morphisms $\mathrm{QHom}(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ is given by

$$\begin{array}{ccc} \sigma^*\mathcal{M}_\eta & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M}_\eta \\ \downarrow & & \downarrow \\ \sigma^*\mathcal{N}_\eta & \xrightarrow{\tau_{\mathcal{N}}} & \mathcal{N}_\eta \end{array}$$

  We denote the resulting category by $\mathcal{M}ot_C^{\underline{\nu}}(S)$.

- Theorem (Analog of Jannsen's semisimplicity result)
  The category $\mathcal{M}ot_C^{\nu}(\overline{\mathbb{F}}_q)$ with the obvious fiber functor ω is a semi-simple tannakian category. In particular the associated motivic group P is pro-reductive.

  Proof.
  cf. [E. and Urs Hartl; Cat of C-motives over finite fields] □

- Theorem (Analog of Jannsen's semisimplicity result)

  The category $\mathcal{M}ot_C^{\nu}(\overline{\mathbb{F}}_q)$ with the obvious fiber functor ω is a semi-simple tannakian category. In particular the associated motivic group P is pro-reductive.

  Proof.

  cf. [E. and Urs Hartl; Cat of C-motives over finite fields] □

- Remark

  For this category one can establish

  -realization functors

  -Tate conjecture

  -analog for Honda-Tate theory and etc...

Still:

-Want to geometrize this cat!

-Want to equip $\mathcal{M}ot_C^{\nu}(S)$ with $\mathfrak{G}$-structure for an affine flat group scheme over C of f.t.!

Still:
-Want to geometrize this cat!
-Want to equip $\mathcal{M}ot_C^{\nu}(S)$ with $\mathfrak{G}$-structure for an affine flat group scheme over C of f.t.!

- Definition (Global $\mathfrak{G}$-shtuka)

  A *global* $\mathfrak{G}$*-shtuka* $\underline{\mathcal{G}}$ over an $\mathbb{F}_q$-scheme S is a tuple $(\mathcal{G}, \underline{s}, \tau)$ consisting of

  - a $\mathfrak{G}$-bundle $\mathcal{G}$ over C_S,
  - an n-tuple $\underline{s} := (s_i) \in C^n(S)$ of (characteristic) sections and
  - an isomorphism $\tau \colon \sigma^*\mathcal{G}|_{C_S \setminus \Gamma_{\underline{s}}} \tilde{\to} \mathcal{G}|_{C_S \setminus \Gamma_{\underline{s}}}$.

  We let $\nabla_n \mathcal{H}^1(C, \mathfrak{G})$ denote the stack whose S-points parameterizes global $\mathfrak{G}$-shtukas over S.

Still:
-Want to geometrize this cat!
-Want to equip $\mathcal{M}ot_C^{\nu}(S)$ with $\mathfrak{G}$-structure for an affine flat group scheme over C of f.t.!

- **Definition (Global $\mathfrak{G}$-shtuka)**

  A *global $\mathfrak{G}$-shtuka* $\underline{\mathcal{G}}$ over an $\mathbb{F}_q$-scheme S is a tuple $(\mathcal{G}, \underline{s}, \tau)$ consisting of

  - a $\mathfrak{G}$-bundle $\mathcal{G}$ over C_S,
  - an n-tuple $\underline{s} := (s_i) \in C^n(S)$ of (characteristic) sections and
  - an isomorphism $\tau \colon \sigma^*\mathcal{G}|_{C_S \setminus \Gamma_{\underline{s}}} \tilde{\to} \mathcal{G}|_{C_S \setminus \Gamma_{\underline{s}}}$.

  We let $\nabla_n\mathcal{H}^1(C, \mathfrak{G})$ denote the stack whose S-points parameterizes global $\mathfrak{G}$-shtukas over S.

- **Theorem**

  $\nabla_n\mathcal{H}^1(C, \mathfrak{G})$ is an ind-DM-stack over C^n which is ind-separated and locally of ind-finite type.

  Proof.

  cf. [E. and Urs Hartl, Uniformization of the moduli stacks of $\mathfrak{G}$-shtukas; theorem 3.15]

- Remark (Functoriality)

- Remark (Functoriality)

  The assignment

$$(C, \mathfrak{G}) \mapsto \nabla_n \mathcal{H}^1(C, \mathfrak{G})$$

  is functorial on $(C, \mathfrak{G})$. In particular $\rho : \mathfrak{G} \to \mathfrak{G}'$ induces

$$\rho_* : \nabla_n \mathcal{H}^1(C, \mathfrak{G}) \to \nabla_n \mathcal{H}^1(C, \mathfrak{G}').$$

- Remark (Functoriality)

  The assignment

$$(C, \mathfrak{G}) \mapsto \nabla_n \mathcal{H}^1(C, \mathfrak{G})$$

is functorial on $(C, \mathfrak{G})$. In particular $\rho : \mathfrak{G} \to \mathfrak{G}'$ induces

$$\rho_* : \nabla_n \mathcal{H}^1(C, \mathfrak{G}) \to \nabla_n \mathcal{H}^1(C, \mathfrak{G}').$$

which is induced by

$$\rho_* : \mathcal{H}^1(C, \mathfrak{G}) \to \mathcal{H}^1(C, \mathfrak{G}'), \ \mathcal{G} \mapsto \mathcal{G} \times^{\mathfrak{G}} \mathfrak{G}'.$$

between the stack of $\mathfrak{G}$-bundles and $\mathfrak{G}'$-bundles over C.

§Realization Functors I

- Definition (Étale realization)

§Realization Functors I

- Definition (Étale realization)

  Fix $\underline{\nu} = (\nu_i) \in C^n$. Let $A_{\underline{\nu}}$ denote the completion of $\mathcal{O}_{C^n}$ at $\underline{\nu}$ and let $\nabla_n \mathcal{H}^1(C, \mathfrak{G})^{\underline{\nu}} = \nabla_n \mathcal{H}^1(C, \mathfrak{G}) \times_{C^n} \operatorname{Spf} A_{\underline{\nu}}$. Assume that S is connected, fix a geometric base point $\overline{s}$ of S.

§Realization Functors I

- Definition (Étale realization)

  Fix $\underline{\nu} = (\nu_i) \in C^n$. Let $A_{\underline{\nu}}$ denote the completion of $\mathcal{O}_{C^n}$ at $\underline{\nu}$ and let $\nabla_n\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}} = \nabla_n\mathcal{H}^1(C,\mathfrak{G}) \times_{C^n} \mathrm{Spf}\, A_{\underline{\nu}}$. Assume that S is connected, fix a geometric base point $\bar{s}$ of S. There is the following étale realization functor

$$\omega^{\underline{\nu}}(-) : \nabla_n\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}}(S) \to Funct^{\otimes}\left(Rep\mathfrak{G}, Mod_{\mathbb{A}^{\underline{\nu}}[\pi^1(S,\bar{s})]}\right)$$

$$\underline{\mathcal{G}} \mapsto \omega^{\underline{\nu}}(\underline{\mathcal{G}}) : \rho \mapsto \varprojlim_{D\subseteq \dot{C}} (\rho_*\mathcal{G}|_{D_{\bar{s}}})^{\tau} \otimes_{\mathbb{O}^{\underline{\nu}}} \mathbb{A}^{\underline{\nu}}$$

§Realization Functors I

- Definition (Étale realization)

  Fix $\underline{\nu} = (\nu_i) \in C^n$. Let $A_{\underline{\nu}}$ denote the completion of $\mathcal{O}_{C^n}$ at $\underline{\nu}$ and let $\nabla_n\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}} = \nabla_n\mathcal{H}^1(C,\mathfrak{G}) \times_{C^n} \operatorname{Spf} A_{\underline{\nu}}$. Assume that S is connected, fix a geometric base point $\bar{s}$ of S. There is the following étale realization functor

$$\omega^{\underline{\nu}}(-) : \nabla_n\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}}(S) \to Funct^{\otimes}\left(Rep\mathfrak{G}, Mod_{\mathbb{A}^{\underline{\nu}}[\pi^1(S,\bar{s})]}\right)$$

$$\underline{\mathcal{G}} \mapsto \omega^{\underline{\nu}}(\underline{\mathcal{G}}) : \rho \mapsto \varprojlim_{D\subseteq \dot{C}} (\rho_*\mathcal{G}|_{D_{\bar{s}}})^{\tau} \otimes_{\mathbb{O}^{\underline{\nu}}} \mathbb{A}^{\underline{\nu}}$$

  Here -$\pi_1(S,\bar{s})$ is the algebraic fundamental group of S.
  -$D_{\bar{s}}$ is finite over $\bar{s} = \operatorname{Spec}\mathbb{F}$ for an algebraically closed field $\mathbb{F}$, and $\rho_*\underline{\mathcal{G}}|_{D_{\bar{s}}}$ is equivalent to (M,τ) where M is a free $\mathcal{O}_{D_{\bar{s}}}$-modules. Then $(\rho_*\underline{\mathcal{G}}|_{D_{\bar{s}}})^{\tau} := \{m \in M : \tau(\sigma^* m) = m\}$ denotes the τ-invariant

§Level Structure

- Definition (H-level structure)

§Level Structure

- **Definition (H-level structure)**

  Assume that $S \in \mathcal{N}ilp_{A_{\underline{\nu}}}$ is connected and fix a geometric point $\bar{s}$ of S. For a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ over S let us consider the set of isomorphisms of tensor functors

$$\mathrm{Isom}^{\otimes}(\omega^{\underline{\nu}}(\underline{\mathcal{G}})(-), \omega^{\circ}(-)),$$

  where $\omega^{\circ}\colon \mathrm{Rep}_{\mathbb{A}^{\underline{\nu}}}\mathfrak{G} \to \mathcal{M}od_{\mathbb{A}^{\underline{\nu}}}$ denote the neutral fiber functor.

§Level Structure

- Definition (H-level structure)

  Assume that $S \in \mathcal{N}ilp_{A_{\underline{\nu}}}$ is connected and fix a geometric point $\bar{s}$ of S. For a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ over S let us consider the set of isomorphisms of tensor functors

  $$\mathrm{Isom}^{\otimes}(\omega^{\underline{\nu}}(\underline{\mathcal{G}})(-), \omega^{\circ}(-)),$$

  where $\omega^{\circ}\colon \mathrm{Rep}_{\mathbb{A}^{\underline{\nu}}} \mathfrak{G} \to \mathcal{M}od_{\mathbb{A}^{\underline{\nu}}}$ denote the neutral fiber functor. The set $\mathrm{Isom}^{\otimes}(\omega^{\underline{\nu}}(\underline{\mathcal{G}})(-), \omega^{\circ}(-))$ admits an action of $\mathfrak{G}(\mathbb{A}^{\underline{\nu}}) \times \pi_1(S, \bar{s})$ where $\mathfrak{G}(\mathbb{A}^{\underline{\nu}})$ acts through $\omega^{\circ}(-)$ by tannakian formalism and $\pi_1(S, \bar{s})$ acts through $\omega^{\underline{\nu}}(\underline{\mathcal{G}})(-)$. For a compact open subgroup $H \subseteq \mathfrak{G}(\mathbb{A}^{\underline{\nu}})$ we define a *rational H-level structure* $\bar{\gamma}$ on a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ over $S \in \mathcal{N}ilp_{A_{\underline{\nu}}}$ to be a $\pi_1(S, \bar{s})$-invariant H-orbit $\bar{\gamma} = H\gamma$ in $\mathrm{Isom}^{\otimes}(\omega^{\underline{\nu}}(\underline{\mathcal{G}})(-), \omega^{\circ}(-))$.

§Realization Functors II

Crystalline realizations

- Definition (Loop groups/ affine flag varieties)

§Realization Functors II

Crystalline realizations

- Definition (Loop groups/ affine flag varieties)
  Let $\mathbb{P}$ be a smooth affine group scheme of finite type over $\mathbb{D} = \operatorname{Spec} k[\![z]\!]$, P the generic fiber of $\mathbb{P}$ over $\dot{\mathbb{D}} := \operatorname{Spec} k(\!(z)\!)$.

§Realization Functors II

Crystalline realizations

- **Definition (Loop groups/ affine flag varieties)**
  Let $\mathbb{P}$ be a smooth affine group scheme of finite type over $\mathbb{D} = \operatorname{Spec} k[\![z]\!]$, P the generic fiber of $\mathbb{P}$ over $\dot{\mathbb{D}} := \operatorname{Spec} k(\!(z)\!)$.

  1. The *group of positive loops (resp. loops) associated with* $\mathbb{P}$

     $$L^+\mathbb{P}(R) := \mathbb{P}(R[\![z]\!]) := \mathbb{P}(\mathbb{D}_R) := \operatorname{Hom}_{\mathbb{D}}(\mathbb{D}_R, \mathbb{P})\,,$$

     $$(resp. LP(R) := P(R(\!(z)\!)) := P(\dot{\mathbb{D}}_R) := \operatorname{Hom}_{\dot{\mathbb{D}}}(\dot{\mathbb{D}}_R, P))\,,$$

     where we write $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$ and $\dot{\mathbb{D}}_R := \operatorname{Spec} R(\!(z)\!)$. It is representable by a scheme (resp. an ind-scheme) of finite type (resp. ind-finite type) over k.

§Realization Functors II

Crystalline realizations

- Definition (Loop groups/ affine flag varieties)

  Let $\mathbb{P}$ be a smooth affine group scheme of finite type over $\mathbb{D} = \operatorname{Spec} k[\![z]\!]$, P the generic fiber of $\mathbb{P}$ over $\dot{\mathbb{D}} := \operatorname{Spec} k(\!(z)\!)$.

  1. The *group of positive loops (resp. loops) associated with* $\mathbb{P}$

  $$L^+\mathbb{P}(R) := \mathbb{P}(R[\![z]\!]) := \mathbb{P}(\mathbb{D}_R) := \operatorname{Hom}_{\mathbb{D}}(\mathbb{D}_R, \mathbb{P}),$$

  $$(resp. LP(R) := P(R(\!(z)\!)) := P(\dot{\mathbb{D}}_R) := \operatorname{Hom}_{\dot{\mathbb{D}}}(\dot{\mathbb{D}}_R, P)),$$

  where we write $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$ and $\dot{\mathbb{D}}_R := \operatorname{Spec} R(\!(z)\!)$. It is representable by a scheme (resp. an ind-scheme) of finite type (resp. ind-finite type) over k.

  2. The affine flag variety $\mathcal{F}\ell_{\mathbb{P}}$ is defined to be the ind-scheme representing the *fpqc*-sheaf associated with the presheaf

  $$R \longmapsto LP(R)/L^+\mathbb{P}(R) = P\left(R(\!(z)\!)\right)/\mathbb{P}\left(R[\![z]\!]\right).$$

  on the category of k-algebras

- Definition (Local $\mathbb{P}$-shtuka)

- Definition (Local $\mathbb{P}$-shtuka)

  a) A *local* $\mathbb{P}$*-shtuka* over $S \in \mathcal{N}ilp_{k[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of

- **Definition (Local $\mathbb{P}$-shtuka)**

  a) A *local* $\mathbb{P}$*-shtuka* over $S \in \mathcal{N}ilp_{k[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of

  - a $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ on S and
  - an isomorphism of the associated loop group torsors $\hat{\tau} \colon \hat{\sigma}^*\mathcal{L} \to \mathcal{L}$.

- Definition (Local $\mathbb{P}$-shtuka)

  a) A *local* $\mathbb{P}$*-shtuka* over $S \in \mathcal{N}ilp_{k[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of

  - a $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ on S and
  - an isomorphism of the associated loop group torsors $\hat{\tau} \colon \hat{\sigma}^*\mathcal{L} \to \mathcal{L}$.
    where $(H^1(S, L^+\mathbb{P}) \to H^1(S, LP),\ \mathcal{L}_+ \mapsto \mathcal{L})$.

- **Definition (Local $\mathbb{P}$-shtuka)**

  a) A *local* $\mathbb{P}$*-shtuka* over $S \in \mathcal{N}ilp_{k[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of

  - a $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ on S and
  - an isomorphism of the associated loop group torsors $\hat{\tau}\colon \hat{\sigma}^*\mathcal{L} \to \mathcal{L}$.
    where $(H^1(S, L^+\mathbb{P}) \to H^1(S, LP),\ \mathcal{L}_+ \mapsto \mathcal{L})$.

  b) A morphism (quasi-isogeny) between $\underline{\mathcal{L}} := (\mathcal{L}_+, \tau) \to \underline{\mathcal{L}}' := (\mathcal{L}'_+, \tau')$ is a commutative diagram

$$\begin{array}{ccc} \sigma^*\mathcal{L} & \xrightarrow{\ \tau\ } & \mathcal{L} \\ \downarrow & & \downarrow \\ \hat{\sigma}^*\mathcal{L}' & \xrightarrow{\ \tau'\ } & \mathcal{L}' \end{array}$$

Definition (Local $\mathbb{P}$-shtuka)

a) A *local* $\mathbb{P}$-*shtuka* over $S \in \mathcal{N}ilp_{k[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of

- a $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ on S and
- an isomorphism of the associated loop group torsors $\hat{\tau}\colon \hat{\sigma}^*\mathcal{L} \to \mathcal{L}$.
  where $(H^1(S, L^+\mathbb{P}) \to H^1(S, LP),\ \mathcal{L}_+ \mapsto \mathcal{L})$.

b) A morphism (quasi-isogeny) between $\underline{\mathcal{L}} := (\mathcal{L}_+, \tau) \to \underline{\mathcal{L}}' := (\mathcal{L}'_+, \tau')$ is a commutative diagram

$$\begin{array}{ccc} \sigma^*\mathcal{L} & \xrightarrow{\tau} & \mathcal{L} \\ \downarrow & & \downarrow \\ \hat{\sigma}^*\mathcal{L}' & \xrightarrow{\tau'} & \mathcal{L}' \end{array}$$

c) We denote the resulting category by $\mathbf{Loc}\mathbb{P}\text{-}\mathbf{Sht}(S)$.

Definition (Crystalline realization functors)

Definition (Crystalline realization functors)

For a place ν on C let $\mathbb{P}_\nu := \mathfrak{G} \times_C \operatorname{Spec} \widehat{\mathcal{O}}_{C,\nu}$ and let P_ν be its generic fiber.

Definition (Crystalline realization functors)

For a place ν on C let $\mathbb{P}_\nu := \mathfrak{G} \times_C \operatorname{Spec} \widehat{\mathcal{O}}_{C,\nu}$ and let P_ν be its generic fiber.
There is a crystalline realization functor

$$\omega_{\nu_i}(-) : \nabla_n \mathcal{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S) \to \mathbf{Loc}\mathbb{P}_{\nu_i}\textbf{-Sht}(S)$$

given by sending $\mathcal{G}$ to its formal completion $\widehat{\mathcal{G}}$ along $\Gamma_{s_i} \subseteq C_S$

Definition (Crystalline realization functors)

For a place ν on C let $\mathbb{P}_\nu := \mathfrak{G} \times_C \operatorname{Spec} \widehat{\mathcal{O}}_{C,\nu}$ and let P_ν be its generic fiber.
There is a crystalline realization functor

$$\omega_{\nu_i}(-) : \nabla_n \mathcal{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S) \to \mathbf{Loc}\mathbb{P}_{\nu_i}\textbf{-Sht}(S)$$

given by sending $\mathcal{G}$ to its formal completion $\widehat{\mathcal{G}}$ along $\Gamma_{s_i} \subseteq C_S$ and then using the following observation

$$\text{Cat of formal } \hat{\mathbb{P}}\text{-torsors}/\mathbb{D}_R \leftrightarrow \text{Cat of } L^+\mathbb{P}\text{-torsors}$$

Here $\hat{\mathbb{P}}$ is the completion of $\mathbb{P}$ at $V(z)$.

§Boundedness Conditions

- Definition (naive definition for BC)

§Boundedness Conditions

- Definition (naive definition for BC)

  Let $\mathbb{P}$ be a smooth affine group scheme over $\mathbb{D}$.

§Boundedness Conditions

- Definition (naive definition for BC)

  Let $\mathbb{P}$ be a smooth affine group scheme over $\mathbb{D}$.

  a) A closed ind-subscheme $\hat{Z}$ of $\widehat{\mathcal{F}\ell}_{\mathbb{P}} := \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_k \operatorname{Spf} k[\![\zeta]\!]$ which is stable under the left L^+P-action, such that $Z := \hat{Z} \times_{\operatorname{Spf} k[\![\zeta]\!]} \operatorname{Spec} k$ is a quasi-compact subscheme of $\mathcal{F}\ell_{\mathbb{P}}$ is called a bound.

§Boundedness Conditions

- Definition (naive definition for BC)

Let $\mathbb{P}$ be a smooth affine group scheme over $\mathbb{D}$.

a) A closed ind-subscheme $\hat{Z}$ of $\widehat{\mathcal{F}\ell_{\mathbb{P}}} := \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_k \operatorname{Spf} k[\![\zeta]\!]$ which is stable under the left L^+P-action, such that $Z := \hat{Z} \times_{\operatorname{Spf} k[\![\zeta]\!]} \operatorname{Spec} k$ is a quasi-compact subscheme of $\mathcal{F}\ell_{\mathbb{P}}$ is called a bound.

b) Let $\widehat{Z}$ be a bound with reflex ring $R_{\widehat{Z}}$. Let $\mathcal{L}_+$ and $\mathcal{L}'_+$ be $L^+\mathbb{P}$-torsors over a scheme S in $\mathcal{N}ilp_{R_{\widehat{Z}}}$ and let $\delta \colon \mathcal{L} \to \mathcal{L}'$ be an isomorphism of the associated LP-torsors. We consider an étale covering $S' \to S$ over which trivializations $\alpha \colon \mathcal{L}_+ \to (L^+\mathbb{P})_{S'}$ and $\alpha' \colon \mathcal{L}'_+ \to (L^+\mathbb{P})_{S'}$ exist. Then the automorphism $\alpha' \circ \delta \circ \alpha^{-1}$ of $(LP)_{S'}$ corresponds to a morphism $S' \to LP \widehat{\times}_k \operatorname{Spf} R_{\widehat{Z}}$. We say that δ is *bounded by* $\widehat{Z}$ if for any such trivialization and for all finite extensions R of $k[\![\zeta]\!]$ over which a representative $\widehat{Z}_R$ of $\widehat{Z}$ exists the induced morphism

$$S' \to \widehat{\mathcal{F}\ell}_{\mathbb{P}}$$

factors through $\widehat{Z}_R$.

$$S' \to \widehat{\mathcal{F}\ell}_{\mathbb{P}}$$

factors through $\widehat{Z}_R$.

c) a local $\mathbb{P}$-shtuka $(\mathcal{L}, \hat{\tau})$ is *bounded by* $\widehat{Z}$ if the isom $\hat{\tau}^{-1}$ is bounded by $\widehat{Z}$. Assume that $\widehat{Z} = \mathcal{S}(\omega) \widehat{\times}_k \operatorname{Spf} k[\![\zeta]\!]$ for a *Schubert variety* $\mathcal{S}(\omega) \subseteq \mathcal{F}\ell_{\mathbb{P}}$, with $\omega \in \widetilde{W}$. Then we say that δ is *bounded by* ω.

$$S' \to \widehat{\mathcal{F}\ell}_{\mathbb{P}}$$

factors through $\widehat{Z}_R$.

c) a local $\mathbb{P}$-shtuka $(\mathcal{L}, \hat{\tau})$ is *bounded by* $\widehat{Z}$ if the isom $\hat{\tau}^{-1}$ is bounded by $\widehat{Z}$. Assume that $\widehat{Z} = \mathcal{S}(\omega) \widehat{\times}_k \operatorname{Spf} k[\![\zeta]\!]$ for a *Schubert variety* $\mathcal{S}(\omega) \subseteq \mathcal{F}\ell_{\mathbb{P}}$, with $\omega \in \widetilde{W}$. Then we say that δ is *bounded by* ω.

- Remark

$$S' \to \widehat{\mathcal{F}\ell}_{\mathbb{P}}$$

factors through $\widehat{Z}_R$.

c) a local $\mathbb{P}$-shtuka $(\mathcal{L}, \hat{\tau})$ is *bounded by* $\widehat{Z}$ if the isom $\hat{\tau}^{-1}$ is bounded by $\widehat{Z}$. Assume that $\widehat{Z} = \mathcal{S}(\omega)\widehat{\times}_k \operatorname{Spf} k[\![\zeta]\!]$ for a *Schubert variety* $\mathcal{S}(\omega) \subseteq \mathcal{F}\ell_{\mathbb{P}}$, with $\omega \in \widetilde{W}$. Then we say that δ is *bounded by* ω.

- Remark

1. The above definition is a naive definition of BC. For the true definition one needs to replace $\hat{Z}$ with an equivalence class $[\hat{Z}]$ of subschemes of $\widehat{\mathcal{F}\ell}_{\mathbb{P},R}$. Here R is a finite extension of discrete valuation rings $k[\![\zeta]\!] \subset R \subset \; k(\!(\zeta)\!)^{\mathrm{alg}}$. The class $[\hat{Z}]$ has a representative over a minimal ring $R_{[\hat{Z}]}$ (called *reflex ring*)

$$S' \to \widehat{\mathcal{F}\ell_{\mathbb{P}}}$$

factors through $\widehat{Z}_R$.

c) a local $\mathbb{P}$-shtuka $(\mathcal{L}, \hat{\tau})$ is *bounded by* $\widehat{Z}$ if the isom $\hat{\tau}^{-1}$ is bounded by $\widehat{Z}$. Assume that $\widehat{Z} = \mathcal{S}(\omega) \widehat{\times}_k \operatorname{Spf} k[\![\zeta]\!]$ for a *Schubert variety* $\mathcal{S}(\omega) \subseteq \mathcal{F}\ell_{\mathbb{P}}$, with $\omega \in \widetilde{W}$. Then we say that δ is *bounded by* ω.

- Remark

1. The above definition is a naive definition of BC. For the true definition one needs to replace $\hat{Z}$ with an equivalence class $[\hat{Z}]$ of subschemes of $\widehat{\mathcal{F}\ell}_{\mathbb{P},R}$. Here R is a finite extension of discrete valuation rings $k[\![\zeta]\!] \subset R \subset \ k(\!(\zeta)\!)^{\mathrm{alg}}$. The class $[\hat{Z}]$ has a representative over a minimal ring $R_{[\hat{Z}]}$ (called *reflex ring*)
2. There is a global version of the BC, which we obtain roughly by replacing $\mathcal{F}\ell_{\mathbb{P}}$ by B-D affine Grassmannian $GR_n(C, \mathfrak{G})$, and $\hat{Z} \subseteq \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$ by global Schubert varieties $\mathcal{Z} \subseteq GR_n(C, \mathfrak{G})$. Then BC $[\mathcal{Z}]$ determines a minimal curve of definition $C_{\mathcal{Z}}$ called *reflex curve*.

§$\nabla\mathcal{H}$-data

In analogy with the Shimura varieties side we define

- Definition ($\nabla\mathcal{H}$-data)

§$\nabla\mathcal{H}$-data

In analogy with the Shimura varieties side we define

- Definition ($\nabla\mathcal{H}$-data)

  a) A $\nabla\mathcal{H}$-datum $(\mathfrak{G}, \underline{\hat{Z}}, H)$ consists of

§$\nabla\mathcal{H}$-data

In analogy with the Shimura varieties side we define

- Definition ($\nabla\mathcal{H}$-data)

a) A $\nabla\mathcal{H}$-datum $(\mathfrak{G}, \underline{\hat{Z}}, H)$ consists of
-a smooth affine group scheme $\mathfrak{G}$ over a sm. proj, curve $C\ /\ \mathbb{F}_q$,

§$\nabla\mathcal{H}$-data

In analogy with the Shimura varieties side we define

- Definition ($\nabla\mathcal{H}$-data)

  a) A $\nabla\mathcal{H}$-datum $(\mathfrak{G}, \underline{\hat{Z}}, H)$ consists of
  -a smooth affine group scheme $\mathfrak{G}$ over a sm. proj, curve $C \ / \ \mathbb{F}_q$,
  -an n-tuple of (local) bounds $\underline{\hat{Z}} := (\hat{Z}_{\nu_i})_{i=1...n}$ at the fixed characteristic places $\nu_i \in C$ and

§$\nabla\mathcal{H}$-data

In analogy with the Shimura varieties side we define

- Definition ($\nabla\mathcal{H}$-data)

  a) A $\nabla\mathcal{H}$-datum $(\mathfrak{G}, \underline{\hat{Z}}, H)$ consists of
  -a smooth affine group scheme $\mathfrak{G}$ over a sm. proj, curve $C\ /\ \mathbb{F}_q$,
  -an n-tuple of (local) bounds $\underline{\hat{Z}} := (\hat{Z}_{\nu_i})_{i=1\ldots n}$ at the fixed characteristic places $\nu_i \in C$ and
  -a compact open subgroup $H \subseteq \mathfrak{G}(\mathbb{A}_C^{\underline{\nu}})$.

§$\nabla\mathcal{H}$-data

In analogy with the Shimura varieties side we define

- Definition ($\nabla\mathcal{H}$-data)

  a) A $\nabla\mathcal{H}$-datum $(\mathfrak{G}, \underline{\hat{Z}}, H)$ consists of
  -a smooth affine group scheme $\mathfrak{G}$ over a sm. proj, curve $C \,/\, \mathbb{F}_q$,
  -an n-tuple of (local) bounds $\underline{\hat{Z}} := (\hat{Z}_{\nu_i})_{i=1\ldots n}$ at the fixed characteristic places $\nu_i \in C$ and
  -a compact open subgroup $H \subseteq \mathfrak{G}(\mathbb{A}_C^{\nu})$.
  b) There is a functorial assignment

$$(\mathfrak{G}, \underline{\hat{Z}}, H) \rightsquigarrow \nabla_n^{\underline{\hat{Z}},H}\mathcal{H}^1(C, \mathfrak{G})$$

where $\nabla_n^{\underline{\hat{Z}},H}\mathcal{H}^1(C, \mathfrak{G})(S)$ parametrizes $(\underline{\mathcal{G}}, \gamma)$ such that $\omega_{\nu_i}(\underline{\mathcal{G}})$ is bounded by $\hat{Z}_{\nu_i}$.

Theorem

$\nabla^{\frac{\hat{Z}}{n},H}\mathcal{H}^1(C,\mathfrak{G})$ *is a formal DM-stack over* $\check{R}_{\underline{\hat{Z}}}$.

Similarly

- Definition (Local $\nabla\mathcal{H}$-data)

  A local $\nabla\mathcal{H}$-datum is a tuple $(\mathbb{P}, \hat{Z}, b)$ consisting of

  - A smooth affine group scheme $\mathbb{P}$ over $\mathbb{D}$ with connected reductive generic fiber P,
  - A local bound $\hat{Z}$,
  - A σ-conjuagacy class of an element $b \in P(\overline{k}((z)))$.

- Definition (R-Z spaces for local $\mathbb{P}$-shtukas)
  Let $\hat{Z} = [\hat{Z}_R]$ be a bound with reflex ring $\check{R}_{\hat{Z}}$.
  Fix a local $\mathbb{P}$-shtuka $\underline{\mathbb{L}}$ over k.

- **Definition (R-Z spaces for local $\mathbb{P}$-shtukas)**

  Let $\hat{Z} = [\hat{Z}_R]$ be a bound with reflex ring $\check{R}_{\hat{Z}}$.
  Fix a local $\mathbb{P}$-shtuka $\underline{\mathbb{L}}$ over k.
  Define *the Rapoport-Zink space for (bounded) local $\mathbb{P}$-shtukas*, as the space given by the following functor of points

$$\begin{aligned}\underline{\check{\mathcal{M}}}_{\underline{\mathbb{L}}}^{\hat{Z}}\colon \quad (\mathcal{N}ilp_{\check{R}_{\hat{Z}}})^o \quad & \to \mathcal{S}ets \\ S \qquad & \mapsto \Big\{ \text{Isomorphism classes of } (\underline{\mathcal{L}}, \delta)\colon \ \text{where:} \\ & \qquad - \underline{\mathcal{L}} \text{ is a local } \mathbb{P}\text{-shtuka} \\ & \qquad\ \ \text{over } S \text{ bounded by } \hat{Z} \text{ and} \\ & \qquad - \overline{\delta}\colon \underline{\mathcal{L}}_{\overline{S}} \to \underline{\mathbb{L}}_{\overline{S}} \text{ a quasi-isogeny} \Big\}.\end{aligned}$$

- Definition (R-Z spaces for local $\mathbb{P}$-shtukas)
  Let $\hat{Z} = [\hat{Z}_R]$ be a bound with reflex ring $\breve{R}_{\hat{Z}}$.
  Fix a local $\mathbb{P}$-shtuka $\underline{\mathbb{L}}$ over k.
  Define *the Rapoport-Zink space for (bounded) local $\mathbb{P}$-shtukas*, as the space given by the following functor of points

$$\begin{aligned} \underline{\breve{\mathcal{M}}}_{\underline{\mathbb{L}}}^{\hat{Z}} : \quad (\mathcal{N}ilp_{\breve{R}_{\hat{Z}}})^o \quad & \to \mathcal{S}ets \\ S \quad & \mapsto \Big\{ \text{Isomorphism classes of } (\underline{\mathcal{L}}, \delta) : \text{ where:} \\ & \qquad - \underline{\mathcal{L}} \text{ is a local } \mathbb{P}\text{-shtuka} \\ & \qquad\ \ \text{over } S \text{ bounded by } \hat{Z} \text{ and} \\ & \qquad - \bar{\delta} \colon \underline{\mathcal{L}}_{\overline{S}} \to \underline{\mathbb{L}}_{\overline{S}} \text{ a quasi-isogeny} \Big\}. \end{aligned}$$

Here $\overline{S} := V(\zeta) \subseteq S$.

- Theorem (Representablity Of R-Z spaces for local $\mathbb{P}$-shtukas)

The functor $\underline{\check{\mathcal{M}}}_{\mathbb{L}}^{\hat{Z}}$ is ind-representable by a formal scheme over Spf $\check{R}_{\hat{Z}}$ *which is locally formally of finite type and separated. It is an ind-closed ind-subscheme of* $\mathcal{F}\ell_{\mathbb{P}}\widehat{\times}_{\mathbb{F}_q}$ Spf $\check{R}_{\hat{Z}}$. *Its underlying reduced subscheme equals a closed ADLV* $X_Z(b)$, *which is a scheme locally of finite type and separated over* $\mathbb{F}$, *all of whose irreducible components are projective.*

Proof.

cf. [E. and Urs Hartl, Local P-sht and their relation... Theorem 4.18] □

▸ Theorem (Representablity Of R-Z spaces for local $\mathbb{P}$-shtukas)

The functor $\underline{\breve{\mathcal{M}}}_{\mathbb{L}}^{\hat{Z}}$ is ind-representable by a formal scheme over $\operatorname{Spf} \breve{R}_{\hat{Z}}$ which is locally formally of finite type and separated. It is an ind-closed ind-subscheme of $\mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}_q} \operatorname{Spf} \breve{R}_{\hat{Z}}$. Its underlying reduced subscheme equals a closed ADLV $X_Z(b)$, which is a scheme locally of finite type and separated over $\mathbb{F}$, all of whose irreducible components are projective.

Proof.

cf. [E. and Urs Hartl, Local P-sht and their relation... Theorem 4.18] □

Definition

The datum $(\mathbb{P}, \hat{Z}, b)$ determines the reflex ring $\breve{R}_{\hat{Z}}$, and a local $\mathbb{P}$-shtuka $\underline{\mathbb{L}} := (L^+\mathbb{P}, b\hat{\sigma})$. This establishes

$$(\mathbb{P}, \hat{Z}, b) \rightsquigarrow \breve{\mathcal{M}}(\mathbb{P}, \hat{Z}, b) := \breve{\mathcal{M}}_{\underline{\mathbb{L}}}^{\hat{Z}}, \qquad (1)$$

which assigns the Rapoport-Zink space $\breve{\mathcal{M}}(\mathbb{P}, \hat{Z}, b) := \underline{\breve{\mathcal{M}}}_{\mathbb{L}}^{\hat{Z}}$ to a local $\nabla\mathcal{H}$-datum $(\mathbb{P}, \hat{Z}, b)$.

- Theorem (Local Model Theorem I)

- **Theorem (Local Model Theorem I)**

  Fix a global $\nabla\mathcal{H}$-datum $(\mathfrak{G}, \underline{\hat{Z}}, H)$. Assume that $\mathfrak{G}$ is smooth over C. Then there is the following roof

$$\begin{array}{ccc} & \nabla_n^{H,\underline{\widehat{Z}}}\widetilde{\mathcal{H}^1}(C,\mathfrak{G})_{R_{\underline{\nu}}} & \\ \swarrow^{\pi} & & \searrow^{\pi^{loc}} \\ \nabla_n^{H,\underline{\widehat{Z}}}\mathcal{H}(C,\mathfrak{G})^1_{R_{\underline{\nu}}} & & \prod_i \widehat{Z}_{\nu_i,R_{\nu_i}}, \end{array} \qquad (2)$$

- Theorem (Local Model Theorem I)

Fix a global $\nabla\mathcal{H}$-datum $(\mathfrak{G}, \underline{\hat{Z}}, H)$. Assume that $\mathfrak{G}$ is smooth over C. Then there is the following roof

$$\begin{array}{ccc} & \nabla_n^{H,\underline{\widehat{Z}}}\widetilde{\mathcal{H}^1}(C,\mathfrak{G})_{R_{\underline{\nu}}} & \\ \swarrow^{\pi} & & \searrow^{\pi^{loc}} \\ \nabla_n^{H,\underline{\widehat{Z}}}\mathcal{H}(C,\mathfrak{G})^1_{R_{\underline{\nu}}} & & \prod_i \widehat{Z}_{\nu_i,R_{\nu_i}}, \end{array} \tag{2}$$

Let y be a geometric point of $\nabla_n^{H,\underline{\widehat{Z}}}\mathcal{H}^1_{R_{\underline{\nu}}}$. The $\prod_i L^+\mathbb{P}_{\nu_i}$-torsor $\pi : \nabla_n^{H,\underline{\widehat{Z}}}\widetilde{\mathcal{H}^1_{R_{\underline{\nu}}}} \to \nabla_n^{H,\underline{\widehat{Z}}}\mathcal{H}^1_{R_{\underline{\nu}}}$ admits a section s, locally over an étale neighborhood of y, such that the composition $\pi^{loc} \circ s$ is formally étale.

- Theorem (Local Model Theorem II)

- **Theorem (Local Model Theorem II)**
  To a local $\nabla\mathcal{H}$-datum $(\mathbb{P}, \hat{Z}, b)$ one can assign a roof

$$\begin{array}{ccc} & \widetilde{\mathcal{M}} & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \pi^{loc}} \\ \check{\mathcal{M}}(\mathbb{P}, \hat{Z}, b) & & \widehat{Z}, \end{array} \tag{3}$$

▸ **Theorem (Local Model Theorem II)**

To a local $\nabla\mathcal{H}$-datum $(\mathbb{P}, \hat{Z}, b)$ one can assign a roof

$$\begin{array}{ccc} & \widetilde{\mathcal{M}} & \\ {}^{\pi}\swarrow & & \searrow^{\pi^{loc}} \\ \check{\mathcal{M}}(\mathbb{P}, \hat{Z}, b) & & \widehat{Z}, \end{array} \tag{3}$$

that satisfies the following properties

1. *the morphism π^{loc} is formally smooth and*
2. *$\widetilde{\mathcal{M}}$ is an $L^{+}\mathbb{P}$-torsor under $\pi : \widetilde{\mathcal{M}} \to \check{\mathcal{M}}(\mathbb{P}, \hat{Z}, b)$. It admits a section s' locally for the étale topology on $\check{\mathcal{M}}(\mathbb{P}, \hat{Z}, b)$ such that $\pi^{loc} \circ s'$ is formally étale.*

Idea of the proof

The proof uses deformation theory of global $\mathfrak{G}$-shtukas. cf. [E. and S. Habibi Loc models for moduli of global G-shukas] and [E. Local model for moduli for local $\mathbb{P}$-shtukas]

Idea of the proof

The proof uses deformation theory of global $\mathfrak{G}$-shtukas. cf. [E. and S. Habibi Loc models for moduli of global G-shukas] and [E. Local model for moduli for local $\mathbb{P}$-shtukas]

- Proposition (Rigidity of quasi-isogenies for local $\mathbb{P}$-shtukas)

  Let S be a scheme in $\mathcal{N}ilp_{k[\![\zeta]\!]}$ and let $j\colon \bar{S} \to S$ be a closed immersion defined by a sheaf of ideals $\mathcal{I}$ which is locally nilpotent. Let $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}'$ be two local $\mathbb{P}$-shtukas over S. Then

$$QIsog_S(\underline{\mathcal{L}}, \underline{\mathcal{L}}') \to QIsog_{\bar{S}}(j^*\underline{\mathcal{L}}, j^*\underline{\mathcal{L}}'), \quad f \mapsto j^* f$$

  is a bijection of sets.

  Proof.

  cf. [E. and Urs Hartl, Local P-sht and their relation...Proposition 2.11].

  □

Let $S \in \mathcal{N}ilp_{A_{\underline{\nu}}}$ and let $j : \overline{S} \to S$ be a closed subscheme defined by a locally nilpotent sheaf of ideals $\mathcal{I}$. Let $\overline{\underline{\mathcal{G}}}$ be a global $\mathfrak{G}$-shtuka $\nabla_n\mathcal{H}^1(C, \mathfrak{G})^{\underline{\nu}}(\bar{S})$. We let $Defo_S(\bar{\underline{\mathcal{G}}})$ denote the category of infinitesimal deformations of $\overline{\underline{\mathcal{G}}}$ over S. More explicitly $Defo_S(\bar{\underline{\mathcal{G}}})$ is the category of lifts of $\overline{\underline{\mathcal{G}}}$ to S, which consists of all pairs $(\underline{\mathcal{G}}, \alpha : j^*\underline{\mathcal{G}} \to \overline{\underline{\mathcal{G}}})$ where $\underline{\mathcal{G}}$ belongs to $\nabla_n\mathcal{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S)$, and α is an isomorphism of global $\mathfrak{G}$-shtukas over S.
Similarly for a local $\mathbb{P}$-shtuka $\bar{\underline{\mathcal{L}}}$ in $Sht_{\mathbb{P}}^{\mathbb{D}}(\overline{S})$ we define the category of lifts $Defo_S(\bar{\underline{\mathcal{L}}})$ of $\bar{\underline{\mathcal{L}}}$ to S.

Let $S \in \mathcal{N}ilp_{A_{\underline{\nu}}}$ and let $j : \overline{S} \to S$ be a closed subscheme defined by a locally nilpotent sheaf of ideals $\mathcal{I}$. Let $\overline{\underline{\mathcal{G}}}$ be a global $\mathfrak{G}$-shtuka $\nabla_n\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}}(\bar{S})$. We let $Defo_S(\bar{\underline{\mathcal{G}}})$ denote the category of infinitesimal deformations of $\overline{\underline{\mathcal{G}}}$ over S. More explicitly $Defo_S(\bar{\underline{\mathcal{G}}})$ is the category of lifts of $\overline{\underline{\mathcal{G}}}$ to S, which consists of all pairs $(\underline{\mathcal{G}}, \alpha : j^*\underline{\mathcal{G}} \to \overline{\underline{\mathcal{G}}})$ where $\underline{\mathcal{G}}$ belongs to $\nabla_n\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}}(S)$, and α is an isomorphism of global $\mathfrak{G}$-shtukas over S.
Similarly for a local $\mathbb{P}$-shtuka $\bar{\underline{\mathcal{L}}}$ in $Sht_{\mathbb{P}}^{\mathbb{D}}(\overline{S})$ we define the category of lifts $Defo_S(\bar{\underline{\mathcal{L}}})$ of $\bar{\underline{\mathcal{L}}}$ to S.

- **Theorem**

  Let $\bar{\underline{\mathcal{G}}} := (\bar{\mathcal{G}}, \bar{\tau})$ be a global $\mathfrak{G}$-shtuka in $\nabla_n\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}}(\bar{S})$. Then the functor

$$Defo_S(\bar{\underline{\mathcal{G}}}) \to \prod_i Defo_S(\omega_{\nu_i}(\bar{\underline{\mathcal{G}}}))\,,$$

  is an equivalence of categories.

  Proof.

  cf. [E. and Hartl, Relation between global and local P-shtukas] □

Number Fields	Function Fields
The group $\mathbb{G}$ over $\mathbb{Q}$	The group $\mathfrak{G}$ over C
characteristic p	characteristic $\underline{\nu} = \{\nu_i\}$
$G_p := \mathbb{G} \times_{\mathbb{Q}} \mathbb{Q}_p$	$\mathbb{P}_{\nu_i}$
$\mathbb{S} \to \mathbb{G}_{\mathbb{R}}$	n-tuple of boundedness conditions $\underline{\hat{Z}}$
A compact open subgroup $K \subseteq \mathbb{G}(\mathbb{A}_{\mathbb{Q}})$	A compact open subgroup $H \subseteq \mathbb{G}(\mathbb{A}_C)$
Shimura data $(\mathbb{G}, X, K)$	$\nabla\mathcal{H}$-data $(\mathfrak{G}, \underline{\hat{Z}}, H)$
reflex ring $\mathcal{O}_E$ of the reflex field $E = E(G, X, K)$	reflex ring $R_{\underline{\hat{Z}}}$
The canonical integral model $\mathcal{S}_K$	Moduli stack $\nabla_n^{H,\underline{\hat{Z}}}\mathcal{H}^1(C, \mathfrak{G})^{\underline{\nu}}$
Local Shimura data $(\mathcal{P}, \{\mu\}, [b])$	Local $\nabla\mathcal{H}$-data $(\mathbb{P}, \hat{Z}, [b])$
p-divisible groups and (iso-)crystals (with additional structure)	Local ($\mathbb{P}$-)Shtukas

Rapoport-Zink space $\check{\mathcal{M}}(\mathcal{P}, \{\mu\}, [b])$ over the reflex ring $\mathcal{O}_{E_\mu}$	Rapoport-Zink space $\check{\mathcal{M}}(\mathbb{P}, \hat{Z}, [b])$ over the reflex ring $R_{\hat{Z}}$
The local model $\mathbf{M}^{loc}$	The scheme $\hat{Z}$
The local Model diagram $\begin{array}{ccc} & \widetilde{\mathcal{M}}(G, \{\mu\}, [b]) & \\ \swarrow^{\pi} & & \searrow^{\pi^{loc}} \\ \check{\mathcal{M}}(G, \{\mu\}, [b]) & & \mathbf{M}^{loc}, \end{array}$	The local Model diagram $\begin{array}{ccc} & \widetilde{\mathcal{M}}(\mathbb{P}, \hat{Z}, [b]) & \\ \swarrow^{\pi} & & \searrow^{\pi^{loc}} \\ \check{\mathcal{M}}(\mathbb{P}, \hat{Z}, [b]) & & \widehat{Z}, \end{array}$
The category of motives $Mot(\overline{\mathbb{F}}_q)$ with realization functors $\omega_\ell(-)$ and $\omega_p(-)$	The category of C-motives $\mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}}_q)$ with realization functors $\omega^{\underline{\nu}}(-)$ and $\omega_{\nu_i}(-)$
fiber functor $\omega(-) : Mot(\mathbb{F}_q) \to \overline{\mathbb{Q}}$-vect. sp. (conjectural)	The fiber functor $\omega : \mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}}_q) \to \overline{Q}$-vect. sp.
Honda-Tate Theory $W(p^\infty)$	Honda-Tate theory $W_{\underline{\nu}}$
(quasi-)motivic galois gerb $\mathfrak{Q}$	The motivic groupoid $\mathfrak{P} := \mathcal{M}ot_C^{\underline{\nu}}(\mathbb{F})(\omega)$

The uniformization map $$\check{\mathcal{M}}(G,\{\mu\},[b]) \times G(\mathbb{A}_f^p)/K$$ $$\Theta\downarrow$$ $$\mathcal{S}_K$$	The uniformization map $$\prod_i \check{\mathcal{M}}(\mathbb{P}_{\nu_i},\widehat{Z}_{\nu_i},b_i) \times \mathfrak{G}(\mathbb{A}_Q^{\nu})/H$$ $$\Theta\downarrow$$ $$\nabla_n^{H,\hat{\underline{Z}}}\mathcal{H}^1(C,\mathfrak{G})^{\underline{\nu}}$$
Kottwitz-Rapoport (resp. Newton) stratification	Kottwitz-Rapoport (resp. Newton) stratification
⋮	⋮

The analogy between Shimura varieties and moduli of G-Shtukas

§Some Applications

§Some Applications

- Flatness, Cohen-Macaulayness and Normality of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathcal{H}^1_{R_{\underline{\nu}}}$ over its reflex ring.

§Some Applications

- Flatness, Cohen-Macaulayness and Normality of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathcal{H}^1_{R_{\underline{\nu}}}$ over its reflex ring.
- ss-trace of Frobenius

$$tr^{ss}(Frob_x; R\Psi_{\underline{x}}^{\nabla^{\hat{Z}}\mathcal{H}}(\overline{\mathbb{Q}}_\ell)) = tr^{ss}(Frob_r; R\Psi_y^{\hat{Z}}\overline{\mathbb{Q}}_\ell).$$

§Some Applications

- Flatness, Cohen-Macaulayness and Normality of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathcal{H}^1_{R_{\underline{\nu}}}$ over its reflex ring.
- ss-trace of Frobenius

$$tr^{ss}(Frob_x; R\Psi_{\underline{x}}^{\nabla^{\hat{Z}}\mathcal{H}}(\overline{\mathbb{Q}}_\ell)) = tr^{ss}(Frob_r; R\Psi_y^{\hat{Z}}\overline{\mathbb{Q}}_\ell).$$

- Kottwitz-Rapoport Stratification of $\nabla\mathcal{H}$:

$$\begin{array}{ccc} & \nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\widetilde{\mathcal{H}^1_{R_{\underline{\nu}}}} & \\ \swarrow^{\pi} & & \searrow^{\pi^{loc}} \\ \nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathcal{H}^1_{R_{\underline{\nu}}} & & \prod_i \widehat{Z}_{\nu_i,R_{\nu_i}}, \end{array} \tag{4}$$

induces a natural stratification $\{(\nabla_n^{H,Z_{\underline{\nu}}}\mathcal{H}^1)^{\underline{\lambda}}\}_{\underline{\lambda}}$. Namely for every algebraically closed field L over $\mathbb{F}_q$ we have

$$\lambda_{\mathfrak{G},\underline{\nu}} : \{(\nabla_n^{H,Z_{\underline{\nu}}}\mathcal{H}_s^1)^{\underline{\lambda}}\}_{\underline{\lambda}} \to |[\prod_{\nu\in\underline{\nu}} L^+\mathbb{P}_\nu\backslash\hat{Z}_\nu]| =: \prod_{\nu\in\underline{\nu}} Adm(\hat{Z}_\nu)$$

$$\subseteq \prod_{\nu} \widetilde{W}_{\mathbb{P}_\nu}.$$

Set $KR_{\underline{\omega}} := \lambda^{-1}_{\mathfrak{G},\underline{\nu}}(\underline{\omega})$. The incidence relation between these strata is given by the obvious partial order on the product $\prod_\nu \widetilde{W}_{\mathbb{P}_\nu}$, induced by the natural Bruhat order.

$$\lambda_{\mathfrak{G},\underline{\nu}} : \{(\nabla_n^{H,Z_{\underline{\nu}}}\mathcal{H}_s^1)^{\underline{\lambda}}\}_{\underline{\lambda}} \to |[\prod_{\nu\in\underline{\nu}} L^+\mathbb{P}_\nu\backslash\hat{Z}_\nu]| =: \prod_{\nu\in\underline{\nu}} Adm(\hat{Z}_\nu)$$

$$\subseteq \prod_\nu \widetilde{W}_{\mathbb{P}_\nu}.$$

Set $KR_{\underline{\omega}} := \lambda_{\mathfrak{G},\underline{\nu}}^{-1}(\underline{\omega})$. The incidence relation between these strata is given by the obvious partial order on the product $\prod_\nu \widetilde{W}_{\mathbb{P}_\nu}$, induced by the natural Bruhat order.

- (IC-cohomology complexes)

  The IC-sheaves $IC(\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathcal{H}_s^1)$ and the restriction of $IC(Hecke_s^{\widehat{Z}_{\underline{\nu}}})$ coincide up to some shift and Tate twists. [E. and Habibi 2019]
  -Recall that the stack $Hecke_n(C,\mathfrak{G})$ and $GR_n \times \mathcal{H}^1(C,\mathfrak{G})$ as families over $C^n \times \mathcal{H}^1(C,\mathfrak{G})$ are locally isomorphic with respect to the étale topology on $C^n \times \mathcal{H}^1(C,\mathfrak{G})$.

$$\lambda_{\mathfrak{G},\underline{\nu}} : \{(\nabla_n^{H,Z_{\underline{\nu}}}\mathcal{H}_s^1)^{\underline{\lambda}}\}_{\underline{\lambda}} \to |[\prod_{\nu\in\underline{\nu}} L^+\mathbb{P}_\nu\backslash\hat{Z}_\nu]| =: \prod_{\nu\in\underline{\nu}} Adm(\hat{Z}_\nu)$$

$$\subseteq \prod_{\nu} \widetilde{W}_{\mathbb{P}_\nu}.$$

Set $KR_{\underline{\omega}} := \lambda_{\mathfrak{G},\underline{\nu}}^{-1}(\underline{\omega})$. The incidence relation between these strata is given by the obvious partial order on the product $\prod_\nu \widetilde{W}_{\mathbb{P}_\nu}$, induced by the natural Bruhat order.

- (IC-cohomology complexes)

  The IC-sheaves $IC(\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathcal{H}_s^1)$ and the restriction of $IC(Hecke_s^{\widehat{Z}_{\underline{\nu}}})$ coincide up to some shift and Tate twists. [E. and Habibi 2019]
  -Recall that the stack $Hecke_n(C,\mathfrak{G})$ and $GR_n \times \mathcal{H}^1(C,\mathfrak{G})$ as families over $C^n \times \mathcal{H}^1(C,\mathfrak{G})$ are locally isomorphic with respect to the étale topology on $C^n \times \mathcal{H}^1(C,\mathfrak{G})$.
- Lang's cycles on $\check{\mathcal{M}}_\eta$

$$\hat{.} \qquad \hat{.}$$

$$|$$

$$\smile$$

Thank you !