Prehomogeneous zeta functions and toric periods for inner forms of $\operatorname{GL}(2)$

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Periods of automorphic representations (1/2)

- G : reductive algebraic group over a number field F
- π : automorphic representation of $G(\mathbb{A}_F)$
- H : subgroup of G

We say that π is H-distinguished if

$$\exists \phi \in \pi \quad \text{ s.t. } \mathcal{P}_{H}(\phi) := \int_{H(F) \setminus H(\mathbb{A}_{F})} \phi(h) dh \neq 0.$$

Haar measure \frown

This integral is called a period with respect to H.

Periods of automorphic representations (2/2)

Periods are closely related to analytic properties of automorphic *L*-functions of π .

In particular, there are many examples like

- \exists non-vanishing periods $\Rightarrow L(1/2, \pi) \neq 0$
- \exists non-vanishing periods $\Leftrightarrow L(s, \pi)$ has a pole

Toric periods (1/2)

- E/F: quadratic extension of number fields
- D: quaternion algebra over F s.t. $E \hookrightarrow D$
- π : cuspidal automorphic representation of $D^{\times}_{\mathbb{A}_{F}}$

We say that π is E^{\times} -distinguished if

$$\exists \phi \in \pi \quad \text{ s.t. } \mathcal{P}_E(\phi) := \int_{E^{\times} \mathbb{A}_F^{\times} \setminus \mathbb{A}_E^{\times}} \phi(h) \mathrm{d}h \neq 0.$$

This integral is called a toric period with respect to E.

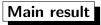
Toric periods (2/2)

• $\chi_E : \mathbb{A}_F^{\times} / F^{\times} \to \{\pm 1\}$: the unique quadratic character s.t. $\operatorname{Ker}(\chi_E) = \operatorname{N}_{E/F}(\mathbb{A}_E^{\times}).$

Then,

$$\pi : E^{\times} \text{-distinguished} \implies L(1/2, \pi \otimes \chi_E) \neq 0.$$
& dim $\pi \neq 1$

This is a reuslt of Waldspurger.



- S : finite set of places of F
- π : irreducible cuspidal automorphic representation

of
$$D^{\times}_{\mathbb{A}}$$
 s.t. dim $\pi \neq 1$

Theorem

Suppose we have $L(1/2, \pi) \neq 0$.

Then $\exists \mathcal{E}_v$: quadratic semi-simple algebra over F_v for $v \in S$ s.t. $\# \left\{ E/F: \text{ quad. ext.} \middle| \begin{array}{c} (1) E \stackrel{\exists}{\hookrightarrow} D \\ (2) E_v = \mathcal{E}_v \quad \forall v \in S \\ (3) \pi \text{ is } E^{\times} \text{-distinguished} \end{array} \right\} = \infty$ The main result is motivated by the remarkable results of Waldspurger in 1980's about:

- Shimura correspondence in the framework of autormorphic representations of Mp_2 ;
- non-vanishing of central *L*-values ;
- non-vanishing of toric periods.

Waldspurger's result (1/3)

For $\xi \in F^{\times}$,

- $E = E_{\xi} = F[X]/(X^2 \xi)$
 - : the associated quadratic algebra over F.
- $\chi_{\xi} = \chi_E : \mathbb{A}_F^{\times} / F^{\times} \to \{\pm 1\}$: the unique quadratic character s.t. $\operatorname{Ker}(\chi_{\xi}) = \operatorname{N}_{E/F}(\mathbb{A}_E^{\times}).$

In particular, $\chi_{\xi} = 1$ iff $E_{\xi} \simeq F \times F$.

Waldspurger's result (2/3)

Suppose that $D = Mat_2(F)$.

- π : irreducible cuspidal automorphic representation of $D^{\times}_{\mathbb{A}_F} = \mathrm{GL}_2(\mathbb{A}_F)$
- $\delta_v \in \mathbb{R}_{>0}$ for each $v \in S$

Theorem (Waldspurger '91) If $\varepsilon(1/2,\pi) = 1$, then

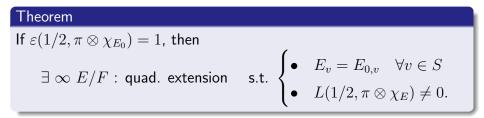
$$\exists \xi \in F^{\times} \text{ s.t. } \begin{cases} \bullet & |\xi - 1|_v < \delta_v \quad \forall v \in S; \\ \bullet & L(1/2, \pi \otimes \chi_{\xi}) \neq 0. \end{cases}$$

Waldspurger's result (3/3)

- This theorem plays an important role in the description of the discrete spectrum of $L^2_{\text{disc}}(\text{Mp}_2(F) \setminus \text{Mp}_2(\mathbb{A}_F))$.
- The original proof is based on the representation theory of $\widetilde{\rm GL}_2$, which was studied by Flicker using the trace formula.
- In 1995, Friedberg and Hoffstein obtained more general results by using the technique of analytic number theory.

result of Friedberg-Hoffstein (1/2)

- E_0 : quadratic extension of F
- π : irreducible cuspidal automorphic representation
 of GL₂(A_F)



result of Friedberg-Hoffstein (2/2)

For the proof, they used a multi-variable Dirichlet series (roughly) of the form :

$$\Phi(s, z, \pi) = \sum_{E} \sum_{\mathfrak{a}} \frac{L(z, \pi \otimes \chi_{E})}{\mathcal{N}(\mathfrak{a})^{s}} \times (\text{some factor}).$$

- E : quadratic extension of F
- \mathfrak{a} : certain ideal class of F.

The above theorem is obtained by analyzing poles at s = 1.

• X : a set

•
$$c(x) \in \mathbb{C}$$
, $x \in X$

Q. How can we prove
$$\#\{x \in X \mid c(x) \neq 0\} = \infty$$
?

A technique of analytic number theory Study the poles of certain Dirichlet series

$$\Phi(X,s) = \sum_{x \in X} \sum_{\mathfrak{a}} \frac{c(x)}{\mathcal{N}(\mathfrak{a})^s} \times (\text{some factor}).$$

typical examples

- Dirichlet's theorem on arithmetic progressions
- Chebotarev's density theorem

The proof of our theorem uses this technique.

Friedberg-Hoffstein vs main theorem

- By using another result of Waldspurger, we can restate Friedberg-Hoffstein as an existence theorem of infinitely many non-vanishing toric periods.
- There is NO obvious implication between Friedberg-Hoffstein and our result in <u>either direction</u>.
- There is an explicit local condition (*) on π and S
 - s.t. Friedberg-Hoffstein + (*) \Rightarrow main theorem

Main tool

The main tool is a **Prehomogeneous zeta function**, roughly of the form

$$Z(s,\phi) = \zeta_F(2s-1) \sum_E \sum_{\mathfrak{a}} \frac{L(1,\chi_E)^2 |\mathcal{P}_E(\phi)|^2}{\mathcal{N}(\mathfrak{a})^s} \times \mathcal{D}_E(s),$$

where
$$\begin{cases} \bullet \quad \phi \in \pi, \\ \bullet \quad \zeta_F : \text{ Dedekind zeta function of } F, \\ \bullet \quad \mathcal{D}_E(s) : \text{ a meromorphic function} \\ & \text{ with a simple pole at } s = 1. \end{cases}$$

Compare the order of the pole at s = 1 of the both sides \rightsquigarrow the sum over E is an infinite sum

Prehomogeneous vector space (1/2)

•
$$G = D^{\times} \times D^{\times} \times \operatorname{GL}_2$$
,

• $V = D \oplus D$

• ρ : $G \curvearrowright V$, F-rational representation, defined by

$$(x, y)\rho(g_1, g_2, g_3) = (g_1^{-1}xg_2, g_1^{-1}yg_2)g_3,$$

 $(g_1, g_2, g_3) \in G, \quad (x, y) \in V.$

This action has a Zariski open orbit V_0 .

$$\stackrel{\text{the triple } (G,V,\rho) \text{ is called a}}{\text{prehomogeneous vector space.}}$$

Prehomogeneous vector space (2/2)

There is a bijection $V^0(F)/G(F) \leftrightarrow X(D)$, where

• L.H.S : the set of F-rational open orbits in V(F)

• R.H.S :
$$X(D) = \left\{ \begin{array}{c} E \\ \end{bmatrix} \quad \begin{array}{c} \text{quad. semi-simple alg.} / F \\ \text{s.t. } E \stackrel{\exists}{\hookrightarrow} D \end{array} \right\}$$

Let $\omega : G \to \mathbb{G}_m$ be the character given by $\omega((g_1, g_2, g_3)) = \det(g_1)^{-2} \det(g_2)^2 \det(g_3)^2$ $(g_1, g_2 \in D^{\times}, g_3 \in \operatorname{GL}_2)$

where det : reduced norm on D or determinant on GL_2

We define the **zeta function** by

$$Z(\Phi, \phi, s) = \int_{G(F)(\operatorname{Ker} \rho)_{\mathbb{A}} \setminus G(\mathbb{A})} |\omega(g)|^{s} \phi(g_{1}) \overline{\phi(g_{2})} \sum_{x \in V^{0}(F)} \Phi(x \cdot \rho(g)) \mathrm{d}g,$$

where

•
$$\phi \in \pi$$
, $g = (g_1, g_2, g_3)$

• Φ : Schwartz-Bruhat function on $V(\mathbb{A})$

Theorem

- (1) The zeta function $Z(\Phi, \phi, s)$ has meromorphic continuation to the whole *s*-plane.
- (2) The zeta function $Z(\Phi, \phi, s)$ satisfies a functional equation $Z(\Phi, \phi, s) = Z^{\vee}(\widehat{\Phi}, \phi, 2 - s),$

where Z^{\vee} is the "dual zeta function" and $\widehat{\Phi}$ is the Fourier transform of Φ .

(3) The possible poles of $Z(\Phi, \phi, s)$ are at s = 1/2 and s = 3/2 and both are at most simple poles.



By the standard unfolding process, we get

$$Z(\Phi,\phi,s) = \frac{1}{2} \sum_{E \in X(D)} \int_{G_{x(E)}(\mathbb{A}) \setminus G(\mathbb{A})} \mathcal{P}_E(\pi(g_1)\phi) \overline{\mathcal{P}_E(\pi(g_2)\phi)} \, |\omega(g)|^s \, \Phi(x(E)\rho(g)) \mathrm{d}g,$$

where

•
$$g = (g_1, g_2, g_3) \in G$$

• x(E) : a fixed representative of the F-rational open orbit which corresponds to $E \in X(D)$

•
$$G_{x(E)}$$
 : the stabilizer of $x(E)$
 $\coloneqq \mathbb{G}_{m,E}$

Intermediate result

As a consequence, we see that

$$\begin{split} Z(\Phi,\phi,s) \not\equiv 0 \\ & \Downarrow \\ \exists \, E \in X(D), \exists \, g \in G \quad \text{s.t.} \quad \mathcal{P}_E(\pi(g)\phi) \neq 0 \\ & \Downarrow \\ \exists \, E \in X(D) \quad \text{s.t.} \quad \pi : E^{\times} \text{-distinguished} \end{split}$$

Theorem

If $L(1/2, \pi) \neq 0$, then

 $\exists \Phi, \exists \phi \in \pi \text{ s.t. } Z(\Phi, \phi, s) \text{ has a simple pole at } s = 1/2.$

In particular, $L(1/2, \pi) \neq 0$ $\Rightarrow \exists \phi \in \pi, \exists E \in X(D) \text{ s.t. } \mathcal{P}_E(\phi) \neq 0$ In order to show non-vanishing of infinitely many periods, we

need an Euler factorization of the contribution of

each *F*-rational open orbit:

$$Z_E(\Phi,\phi,s)$$

:= $\frac{1}{2} \int_{G_{x(E)}(\mathbb{A})\setminus G(\mathbb{A})} \mathcal{P}_E(\pi(g_1)\phi) \overline{\mathcal{P}_E(\pi(g_2)\phi)} |\omega(g)|^s \Phi(x(E)\rho(g)) \mathrm{d}g.$

$$\checkmark \qquad$$
 we have $Z(\Phi, \phi, s) = \sum_{E \in X(D)} Z_E(\Phi, \phi, s).$

We want a factorization of $Z_E(\Phi, \phi, s)$.

Waldspurger formula

For each place v, we can take $\alpha_{E_v} \in \operatorname{Hom}_{E_v^{\times} \times E_v^{\times}}(\pi_v \boxtimes \overline{\pi}_v, \mathbb{C}^{\times})$

(an $E_v^{\times} \times E_v^{\times}$ -invariant hermitian pairing)

so that we have the following Euler factorization:

Theorem (Waldspurger'85)

For a factorizable $\phi = \otimes_v \phi_v$, we have

$$|\mathcal{P}_E(\phi)|^2 = \frac{1}{4} \cdot \frac{\zeta_F(2)L(1/2,\pi)L(1/2,\pi\otimes\chi_E)}{L(1,\pi,\mathrm{Ad})L(1,\chi_E)} \prod_v \alpha_{E_v}(\phi_v,\phi_v).$$

Euler factorization

Applying this formula, we get

$$Z_E(\Phi,\phi,s) = \frac{(L\text{-values})}{(\text{constant})} \times \prod_v Z_{E,v}(\Phi_v,\phi_v,s)$$

for $\Phi = \otimes_v \Phi_v$ and $\phi = \otimes_v \phi_v$.

Here, $Z_{E_v}(\Phi_v, \phi_v, s)$ is the **local zeta function** given by

 $\int_{G_{x(E)}(F_v)\backslash G(F_v)} \alpha_{E_v}(\pi_v(g_1)\phi_v,\pi_v(g_2)\phi_v) |\omega_v(g)|^{s-2} \Phi_v(x(E)\rho(g)) dg$

 \times (constant)(local *L*-values)

Explicit formula

Computing the local zeta functions at unramified places, we see that

$$\begin{split} Z(\Phi,\phi,s) &= \\ \sum_{\mathcal{E}_S \in X(D_S)} \left(\prod_{v \in S} (\mathsf{local zeta})_v \right) \sum_{E \in X(D,\mathcal{E}_S)} \frac{L(1,\chi_E)^2 |\mathcal{P}_E(\phi)|^2}{\mathrm{N}(\mathfrak{a}_E)^{s-1}} \cdot \mathcal{D}_E^S(\pi,s) \\ &\times \zeta_F^S(2s-1) \cdot (\mathsf{constant})(L\text{-values}) \end{split}$$

where

- $\mathfrak{a}_E \subset F$: a certain ideal
- $\mathcal{D}_E^S(\pi, s)$ is a meromorphic function.

Proof of the main theorem (1/2)

Assume the sum over $X(D, \mathcal{E}_S)$ is a non-empty finite sum.

Then,

• $Z(\Phi, \phi, s)$ is holomorphic at s = 1.

 \leadsto L.H.S is holomorphic at s=1

- $\zeta_F^S(2s-1)$ has a simple pole at s=1.
- $\mathcal{D}_E^S(\pi, s)$ has a simple pole at s = 1.

 \rightsquigarrow **R.H.S** has a pole of order 2 at s = 1.

This is a contradiction.

 \Rightarrow The sum over $X(D, \mathcal{E}_S)$ is empty or an infinite sum.

 \Rightarrow If $\exists E \in X(D, \mathcal{E}_S)$ s.t. $\mathcal{P}_E(\phi) \neq 0$, then $\exists \infty$ such E.

Proof of the main theorem (2/2)

Suppose $L(1/2, \pi) \neq 0$.

Then, by the previous theorem, $\exists \phi \in \pi, \exists E_0 \in X(D)$ s.t. $\mathcal{P}_{E_0}(\phi) \neq 0$.

Set $\mathcal{E}_S := \prod_{v \in S} E_{0,v}$.

From the above argument, $\exists \infty E \in X(D, \mathcal{E}_S)$ s.t. $\mathcal{P}_E(\phi) \neq 0$.

This completes the proof the main theorem.

In the main theorem, we do not have a control on on the local components \mathcal{E}_S of the quad. alg. at 'bad places'.

This is the reason why we did not obtain a generalization of Friedberg-Hoffstein.

What is required for extending the main theorem to cover Friedberg-Hoffstein is a close study on the functional equations of local zeta functions

at 'bad places'.

Future works (2/4)

• F. Sato '06 \cdots proved the local F.E. for some specific representations of $\operatorname{GL}_2(\mathbb{R})$ in our setting and computed the 'gamma factors' explicitly.

• Wen-Wei Li '18 ··· proved the local F.E. in general case.

\rightsquigarrow We get the following partial result:

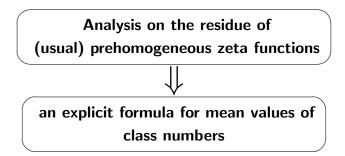
Theorem

Suppose that
$$F = \mathbb{Q}$$
 and $D_{\infty} = \operatorname{Mat}_2(\mathbb{R})$.

Assume that
$$L(1/2, \pi) \neq 0$$
.

Then, $\exists \infty E$: real quad. fields s.t. π is E^{\times} -distinguished.

 \exists another direction to refine the main theorem



Apply this technique to $Z(s, \Phi, \phi)$, to get a <u>quantitative result</u> on non-vanishing of toric periods.

Future works (4/4)

Suppose that $\sum_{E \in X(D, \mathcal{E}_S)} \frac{L(1, \chi_E)^2 |\mathcal{P}_E(\phi)|^2}{\mathrm{N}(\mathfrak{a}_E)^{s-1}} \cdot \mathcal{D}_E^S(\pi, s) \text{ has a simple pole at } s = 3/2.$

Applying a theorem of Tauberian type, we might be able to get the following density theorem:

$$\begin{array}{c} \hline & \text{Density Theorem} \\ \\ \lim_{t \to \infty} t^{-\frac{1}{2}} \sum_{\substack{E \in X(D, \mathcal{E}_S) \\ N(\mathfrak{a}_E) \leq t}} L(1, \chi_E)^2 |\mathcal{P}_E(\phi)|^2 = \begin{pmatrix} \text{the residue of} \\ \text{the above series} \\ \text{at } s = 3/2 \end{pmatrix} \\ \\ \\ \text{R.H.S will be written as a Godement-Jacquet zeta integral.} \end{array}$$

Thank you for your attention.