Stable Trace Formula for Metaplectic Groups

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The cover picture is taken from the Oakland Museum of California. Photographer: Lonnie Wilson. Date: July 4, 1968.

An incomplete list of works mentioned in this talk.

- J. Arthur, *A stable trace formula I—III*. (2002, 2001, 2003).
- J. Arthur, *The endoscopic classification of representations*, AMS Coll. Volume 61 (2013).
- W. T. Gan, A. Ichino, *The Shimura–Waldspurger correspondence for* Mp_{2n} (2018).
- L., *Transfert d'intégrales orbitales pour le groupe métaplectique* (2011)
- L., *La formule des traces stable pour le groupe métaplectique: les termes elliptiques* (2015)
- C. Luo, *Endoscopic character identities for metaplectic groups*, to appear in Crelle's Journal. arXiv:1801.10302
- C. Mœglin, J.-L. Waldspurger, *Stabilisation de la formule des traces tordue, Volume I, II*. Progress in Mathematics, 316—317 (2016).

What is the Arthur--Selberg trace formula?

- F: number field[1], $\mathbb{A} = \mathbb{A}_F$: its ring of adèles.
- G: connected reductive F-group, such as $\mathrm{GL}(n)$.
- $L^2(G(F)\backslash G(\mathbb{A})^1) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}}$: the L^2-automorphic spectrum, $\mathrm{mes}(G(F)\backslash G(\mathbb{A})^1) < +\infty$.

Study of automorphic representations $\approx$ decomposition of $L^2(G(F)\backslash G(\mathbb{A})^1)$ under right regular $G(\mathbb{A})$-representation.

Example – Arthur's Conjecture: $L^2(G(F)\backslash G(\mathbb{A})^1) = \bigoplus_\psi L^2_\psi$, where

- ψ ranges over Arthur parameters $\mathscr{L}_F \times \mathrm{SL}(2, \mathbb{C}) \to {}^{\mathrm{L}}G$,
- $\mathscr{L}_F$ is the hypothetical Langlands group of F.

[1] We exclude the important and interesting case of function fields

Idea of trace formula: access $L^2(G(F)\backslash G(\mathbb{A})^1)$ through an equality of two distributions on $G(\mathbb{A})$.

$$I^G_{\mathrm{geom}}(f) = I^G_{\mathrm{spec}}(f).$$

Spectral side Main terms: sums of character-distributions $f \mapsto \operatorname{tr}\pi(f)$ where π are unitary irreducible representations of $G(\mathbb{A})$, weighted by their multiplicities $m(\pi)$ in L^2_{disc}.

Geometric side Main terms: sums of orbital integrals

$$f \mapsto \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\,\mathrm{d}g,$$

weighted by $\mathrm{mes}\left(G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1\right)$, where γ are elliptic regular semisimple orbits in $G(F)$ and $G_\gamma := Z_G(\gamma)^\circ$.

Example: Comparison of geometric sides for different groups $\rightsquigarrow$ cases of Langlands' Functoriality.

Structure of the trace formula

Difficulty: Non-compactness of $G(F)\backslash G(\mathbb{A})^1 \iff$ Existence of proper Levi subgroups $\iff$ Continuous spectrum in L^2.

Arthur's invariant trace formula

$$I^G = \sum_{\substack{M \supset M_0 \\ Levi}} \frac{|W_0^M|}{|W_0^G|} I_M^G, \quad I \in \{I_{\text{geom}}, I_{\text{spec}}\}$$

- M_0: a fixed minimal Levi of G,
- W_0^M: the Weyl group relative to $M_0 \subset M$,
- I_M^G: invariant distribution with an expansion indexed by γ (resp. π) in M.

Based by truncation + many, many other tools.

Dramatis personae

Terms of local nature in I_M^G: Let f be a test function on $G(\mathbb{A})$.

- $I_M^G(\gamma, f)$: the *invariant version* of weighted orbital integrals, where γ: conjugacy classes in M,
- $I_M^G(\pi, f)$: the *invariant version* of weighted characters, where π: unitary representation of M.

When $G = M$, we recover the usual orbital integrals and characters. The terms $G = M$ contains the "main terms" we want + "other shadows".

Known applications

1. Jacquet–Langlands correspondence.
2. Cyclic base change / automorphic induction for $\mathrm{GL}(n)$.
3. Endoscopic classification of representations for classical groups.

The last case requires a stable trace formula and its twisted analogue (Arthur, Mœglin–Waldspurger, ...), based on (twisted) *Endoscopy* by Langlands–Shelstad–Kottwitz.

$$I^G(f) = \sum_{\substack{\mathbf{G}' \\ \text{ell. endo. data}}} \iota(G, \mathbf{G}') S^{G'}(f'),$$

- $S^{G'}$: stable distributions on the endoscopic group G' (quasisplit), defined recursively;
- $\iota(G, \mathbf{G}') \in \mathbb{Q}_{>0}$: explicit coefficients;
- $f \mapsto f'$: transfer of test functions from G to G' (of a local nature).

The metaplectic cover

Let $\mathrm{Sp}(2n) \subset \mathrm{GL}(2n)$ be the symplectic group. Let $\mu_m = \{z \in \mathbb{C}^\times : z^m = 1\}$. The global metaplectic covering is a central extension of locally compact groups

$$1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(2n, \mathbb{A}) \to \mathrm{Sp}(2n, \mathbb{A}) \to 1.$$

- There is a canonical splitting over $\mathrm{Sp}(2n, F)$.
- It depends on the choice of a symplectic space of dimension $2n$ and an additive character $\psi : F \backslash \mathbb{A} \to \mathbb{C}^\times$.
- It is the restricted product of local coverings $1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(2n)_v \to \mathrm{Sp}(2n, F_v) \to 1$, modulo $\{(z_v)_v \in \bigoplus_v \mu_8 : \prod_v z_v = 1\}$.
- Can be reduced to a central extension by μ_2, but I opt for the *eightfold way*.

1. We are interested in studying genuine representations and automorphic forms of $\widetilde{\mathrm{Sp}}(2n)$, i.e. on which μ_8 acts by $z \mapsto z \cdot \mathrm{id}$.
2. Relevance of metaplectic covering: Θ-lifting, Θ-functions, etc.
3. The genuine representation theory of $\widetilde{\mathrm{Sp}}(2n)$ (both local and global) are largely elucidated by Gan–Savin, Gan–Ichino.
4. A model for *Langlands' program for covering group* (Weissman), which considers *Brylinski–Deligne extensions* of connected reductive groups.
5. Other Brylinski–Deligne coverings occurring naturally:
    - coverings of $\mathrm{GL}(n)$ (Kazhdan–Patterson),
    - higher coverings of symplectic groups (Friedberg, Ginzburg *et al.*)

Key feature of $\widetilde{\mathrm{Sp}}(2n)$: two elements $\tilde{\delta}, \tilde{\delta}'$ commute in $\widetilde{\mathrm{Sp}}(2n)_v$ iff their images $\delta, \delta' \in \mathrm{Sp}(2n, F_v)$ commute.

Invariant trace formula for coverings

Most results in harmonic analysis extends to covering. The invariant trace formula à la Arthur

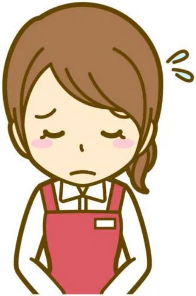
$$I^{\tilde{G}} = \sum_{M} \frac{|W_0^M|}{|W_0^G|} I_{\tilde{M}}^{\tilde{G}}$$

is known under the following technical assumptions.

- *Satake isomorphism* at the unramified places (OK for BD-coverings),
- *Trace Paley–Wiener theorem* for K-finite functions at Archimedean places (OK for $\widetilde{\mathrm{Sp}}(2n)$ and its Levi).

What remains is a stabilization à la Arthur. This requires a theory of endoscopy for coverings.

We can only give a quick sketch of relevant definitions and results. The full stabilization is still a WORK IN PROGRESS.



Endoscopy for $\widetilde{\mathrm{Sp}}(2n)$

Let $\tilde{G} = \widetilde{\mathrm{Sp}}(2n)$, $G = \mathrm{Sp}(2n)$. In both local and global cases:

- Dual group: $\tilde{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$ with trivial Galois action.
- Elliptic endoscopic data $\mathbf{G}^! \leftrightarrow$ pairs $(n', n'') \in \mathbb{Z}^2_{\geq 0}$ such that $n' + n'' = n$. No symmetry here!
- Endoscopic group associated with $\mathbf{G}^!$:
  $G^! = \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$, split.
- Can define
  - a correspondence of stable semisimple conjugacy classes,
  - the factors $\iota(\tilde{G}, \mathbf{G}^!)$ as before,
  - transfer factors Δ, hence a notion of transfer.

Note. Over every Levi $\prod_i \mathrm{GL}(n_i) \times \mathrm{Sp}(2m)$ of G, the 8-fold covering splits canonically into $\prod_i \mathrm{GL}(n_i, F) \times \widetilde{\mathrm{Sp}}(2m)$.

Known results

To study genuine representations, we consider anti-genuine test functions[2] on $\tilde{G}$, on which the characters and orbital integrals are evaluated. Denote this space as $C^\infty_{c,--}(\tilde{G})$.

1. Locally, for each endoscopic datum $\mathbf{G}^!$ we have the transfer of orbital integrals $f \leftrightarrow f^!$.
2. In the unramified local case, we have:
   - Fundamental Lemma for units: the unit of $\mathscr{H}_{--}(K\backslash\tilde{G}/K)$ to that of $\mathscr{H}(K^!\backslash G^!(F)/K)$, where $K = G(\mathfrak{o})$, $K^! = G^!(\mathfrak{o})$.
   - Fundamental Lemma in general: transfer via a canonical $b : \mathscr{H}_{--}(K\backslash\tilde{G}/K) \to \mathscr{H}(K^!\backslash G^!(F)/K)$ (Caihua Luo).
   - Weighted Fundamental Lemma
3. Stabilization of the elliptic semisimple terms in $I^{\tilde{G}}_{\mathrm{geom}}$ is known.

[2] i.e. $f(z\tilde{x}) = z^{-1} f(\tilde{x})$ for all $z \in \mu_8$.

The hoped-for stable trace formula

Hoped-for Theorem (ongoing)

Consider the global covering $\tilde{G} \twoheadrightarrow G(\mathbb{A})$. For every $f = \prod_v f_v \in C^\infty_{c,--}(\tilde{G})$, we expect an identity

$$I^{\tilde{G}}(f) = \sum_{\mathbf{G}^!:\text{ell. endo. data}} \iota(\tilde{G}, \mathbf{G}^!) S^{G^!}(f^!),$$

where

- $f^! = \prod_v f^!_v$ is a transfer of f to $G^!(\mathbb{A})$,
- $S^{G^!}$ is the stable distribution obtained in Arthur's stabilization.

Note that the spectral expansion of $S^{G^!}$ is known (the *stable multiplicity formula* of Arthur for split odd SO).

Potential applications

Following Arthur, we expect

$$I^{\tilde{G}}_{\mathrm{disc}}(f) = \sum_{\mathbf{G}^!} \iota(\tilde{G}, \mathbf{G}^!) S^{G^!}_{\mathrm{disc}}(f^!).$$

This should yield information about the automorphic spectrum of $\tilde{G}$, as well as local information: LLC for local $\widetilde{\mathrm{Sp}}(2n)$ + endoscopic character relations.

- The LLC is known via Θ-correspondence (Gan–Savin); its compatibility with endoscopic character relations is verified by Caihua Luo.
- Using Θ-lifting, Gan and Ichino already obtained a multiplicity formula for the *tempered automorphic spectrum*, fitting into Arthur's conjecture.
- If successful, the stable trace formula should be able to deal with the whole $L^2_{\mathrm{disc,genuine}}(G(F)\backslash\tilde{G})$.

The local geometric statement

Consider the metaplectic covering $1 \to \mu_8 \to \tilde{G} \to G(F) \to 1$ where F is local, $\mathrm{char}(F) = 0$.

Local Geometric Theorem (ongoing)

Let $M \subset G$ be a Levi, $\tilde{\gamma}$ an $M(F)$-conjugacy class in $\tilde{M}$ (more generally, a "geometric" invariant distribution), then

$$I_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, f) = I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\tilde{\gamma}, f)$$

for all anti-genuine f.

Here, $I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\tilde{\gamma}, \cdot)$ is the endoscopic avatar of the geometric distribution $I_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, \cdot)$ in the invariant trace formula for $\tilde{G}$.

Note. There is also an unramified version which reduces to the Weighted Fundamental Lemma.

$$I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\mathbf{M}^!, \delta, f) = \sum_s i_{M^!}(\tilde{G}, G^![s]) S_{M^!}^{G^![s]}\left(\delta[s], B, f^{G^![s]}\right),$$

where s indexes diagrams

$$\begin{array}{ccc} G^![s] & \underset{\text{endo.}}{\overset{\text{ell.}}{\leftarrow\!-\!-\!\rightarrow}} & \tilde{G} \\ {\scriptstyle \text{Levi}}\big\uparrow & & \big\uparrow{\scriptstyle \text{Levi}} \\ M^! & \underset{\text{endo.}}{\overset{\text{ell.}}{\leftarrow\!-\!-\!\rightarrow}} & \tilde{M} \end{array}$$

- δ is a stable geometric distribution $M^!(F)$,
- $i_{M^!}(\tilde{G}, G^![s])$ are explicit constants defined by dual groups,
- $S_{M^!}^{G^![s]}(\cdots)$ are the stable distributions from Arthur,
- $\delta \mapsto \delta[s]$ is a twist by some central element $z[s] \in M^!(F)$. A *metaplectic feature*!

B-functions

The B above prescribes an adjustment of root-lengths in $M^!_\delta$ and $G^!_{\delta[s]}$.
Here: type $\mathrm{B}_n \leftrightarrow \mathrm{C}_m$.

- It affects the definition of weighted orbital integrals (Mœglin–Waldspurger).
- Occurring only in twisted endoscopy and in the metaplectic case. The same rescaling occurs also in the formation of ${}^{\mathrm{L}}\tilde{G}$.
- It fades away when we pass to the global setting.

One shows that $I^{\tilde{G},\mathscr{E}}_{\tilde{M}}(\mathbf{M}^!, \delta, f)$ depends only on the transfer of δ to $\tilde{M}$.
This defines $I^{\tilde{G},\mathscr{E}}_{\tilde{M}}(\tilde{\gamma}, f)$. When $G = M$ and γ is regular, we recover the *transfer of orbital integrals*.

Reduction to G-regular case

- F non-Archimedean: can be done by descent + Shalika germs + known results from Arthur and Mœglin–Waldspurger (nonstandard endoscopy).
- F Archimedean: more difficult — a subtle analysis of the maps ρ_J, σ_J defined à la Mœglin–Waldspurger.

In our case, we also have to consider coverings of the form

$$1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(2a) \overset{\mu_8}{\times} \widetilde{\mathrm{Sp}}(2b) \to \mathrm{Sp}(2a, F) \times \mathrm{Sp}(2b, F) \to 1$$

where $F = \mathbb{R}$ or $\mathbb{C}$.

Global descent

Let F be a number field. Following Mœglin–Waldspurger, the terms with $M = G$ in $I^{\tilde{G}}_{\mathrm{geom}}(f)$ are grouped into

$$A^{\tilde{G}}(V, \mathscr{O}) : \text{ a “geometric” distribution on } \tilde{G}_V,$$

- $\mathscr{O}$: elliptic semisimple conjugacy class in $G(F)$,
- V: finite set of places, sufficiently large w.r.t. $\mathscr{O}$ (otherwise set $A^{\tilde{G}}(V, \mathscr{O}) = 0$).

It can be reduced to the unipotent coefficients (i.e. $\mathscr{O} = \{1\}$) of G_γ, where $\gamma \in \mathscr{O}$.
Goal: Stabilize these coefficients!.

Let $\mathscr{X}$ be a stable elliptic semisimple conjugacy class in $G(F)$. Set

$$A^{\tilde{G}}(V, \mathscr{X}) = \sum_{\mathscr{O} \subset \mathscr{X}} A^{\tilde{G}}(V, \mathscr{O}).$$

This is a finite sum.

Hoped-for Theorem (ongoing)

$$A^{\tilde{G}}(V, \mathscr{X}) = \sum_{\mathbf{G}^!\text{:ell. endo.}} \sum_{\substack{\mathscr{X}^! \\ \mathscr{X}^! \mapsto \mathscr{X}}} \iota(\tilde{G}, \mathbf{G}^!) \underbrace{SA^{G^!}(V, \mathscr{X}^!)}_{\text{stable avatar}}.$$

This generalizes the stabilization of elliptic terms in $I^{\tilde{G}}_{\mathrm{geom}}$. Ingredients:

- Known results concerning various $A^{G_\gamma}_{\mathrm{unip}}(\cdots)$.
- Manipulation of non-abelian Galois cohomologies.

Cancellation of singularities

Local goal

Let F be local, $M \subset G$ be a Levi, and $\tilde{\gamma} \in \tilde{M}$ be G-regular. Wanted:

$$I_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, f) = I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\tilde{\gamma}, f)$$

for all anti-genuine test function f on $\tilde{G}$.

Fact

There exists $\epsilon_{\tilde{M}}$, mapping f to a cuspidal anti-genuine test function on $\tilde{M}$, whose usual orbital integral satisfies

$$I^{\tilde{M}}(\tilde{\gamma}, \epsilon_{\tilde{M}}(f)) = I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\tilde{\gamma}, f) - I_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, f).$$

This requires a lot of analytic arguments. For Archimedean F, we also have to normalize the intertwining operators, stabilize the *differential equations* and *jump conditions* of weighted orbital integrals.

The final touch

The equality $\epsilon_{\tilde{M}}(f) = 0$ will be established altogether with

- induction hypotheses (for smaller G or larger M),
- the global stabilization $I^{\tilde{G}}(f) = \sum_{\mathbf{G}^!} \iota(\tilde{G}, \mathbf{G}^!) S^{G^!}(f^!)$,
- the stabilization of local trace formula for $\tilde{G}$.

Fix an elliptic endoscopic datum $\mathbf{M}^!$ for $\tilde{M}$. Define

$$\epsilon_{\tilde{M}}^{\mathbf{M}^!}(f)(\delta) := \sum_{\gamma} \Delta(\delta, \tilde{\gamma}) \underbrace{I^{\tilde{M}}(\tilde{\gamma}, \epsilon_{\tilde{M}}(f))}_{\text{usual orbital integral}}$$

for all stable regular class δ in $M^!(F)$.

Let $f^{\mathbf{M}^!}_{\tilde{M}}$ be the transfer of the parabolic descent $f_{\tilde{M}}$ of f to $M^!$. View it as a function on regular semisimple stable classes, by taking its stable orbital integrals.

Key local hypothesis

There is a smooth function $\epsilon(\mathbf{M}^!, \cdot)$ on $M^!_{M\text{-reg}}(F)$ such that

$$\epsilon^{\mathbf{M}^!}_{\tilde{M}}(f)(\delta) = \epsilon(\mathbf{M}^!, \delta) f^{\mathbf{M}^!}_{\tilde{M}}(\delta) \quad \text{for all } f, \delta.$$

This is established by a local–global argument, using the spectral side of the global trace formula.

Lemma (ongoing)

We have $\epsilon(\mathbf{M}^!, \delta) + \overline{\epsilon(\mathbf{M}^!, \delta)} = 0$ for all f, $\mathbf{M}^!$ and δ.

The proof is based on the local trace formula.

Since $\epsilon_{\tilde{M}}(f)$ can be recovered from various $\epsilon_{\tilde{M}}^{\mathbf{M}^!}(f)$ by the adjoint transfer / Fourier inversion, the local goal is reduced to the

Lemma (ongoing)

We have $\epsilon(\mathbf{M}^!, \delta) = \overline{\epsilon(\mathbf{M}^!, \delta)}$ for all f, $\mathbf{M}^!$ and δ.

- Roughly speaking, it boils down to showing that the endoscopic transfer is "isomorphic to its complex conjugate".
- In the metaplectic case, complex conjugation can be realized by replacing $\psi \circ \langle \cdot | \cdot \rangle$ by its inverse, where $(W, \langle \cdot | \cdot \rangle)$ is the symplectic vector space giving rise to G.
- This can be achieved by the MVW-involution from $\tilde{\mathrm{G}}$ to its opposite covering, realized by $\mathrm{Ad}(g)$ with $g \in \mathrm{GSp}(W)$ such that $\nu(g) = -1$.

In the uncovered case and its twisted analogue, the *Chevalley involution* is used by Arthur and Mœglin–Waldspurger, in a slightly different way.

Thank you!