SIMONS SYMPOSIA 2016

Geometric Aspects of the Trace Formula

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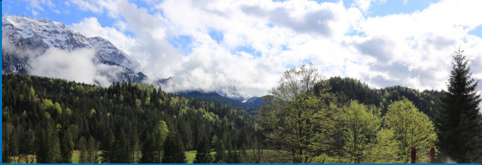

Remarks on the Gan-Ichino multiplicity formula

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The cover picture is taken from the website
`http://www.schloss-elmau.de` .

Main references

- Gan's talk in this symposium.
- Arthur, *Endoscopic classifications of representations: Orthogonal and Symplectic Groups*.
- L. ($>$ 2016).

Today's talk: just some speculations.

Metaplectic groups

Let F be a local field of characteristic 0. The metaplectic covering is a central extension

$$1 \to \mu \to \mathrm{Mp}(2n, F) \xrightarrow{\mathbf{p}} \mathrm{Sp}(2n, F) \to 1$$

with $\mu \subset \mathbb{C}^\times$ finite and $\mathbf{p}$ continuous.

- More precisely, we work with a $2n$-dimensional symplectic F-vector space and $\psi_F : F \to \mathbb{C}^\times$.
- Usually take $\mu = \{\pm 1\}$. To use the Schrödinger model, it is convenient to enlarge μ to μ_8.
- With $\mu = \mu_8$, the Levi subgroups are identifiable with $\prod_i \mathrm{GL}(n_i) \times \mathrm{Mp}(2m, F)$ with $\sum_i n_i + m = n$.

There is a canonical central element above $-1 \in \mathrm{Sp}(2n, F)$, denoted again by -1.

For number fields, we still have $\mathrm{Mp}(2n, \mathbb{A}_F)$. Relevance of $\tilde{G} := \mathrm{Mp}(2n, F)$:

1. Siegel modular forms of half-integral weight.
2. θ-correspondence.
3. Symplectic groups and $\tilde{G}$ are coupled in automorphic descent (Fourier-Jacobi coefficients).
4. It is the most accessible family of covering groups with nontrivial notions of *stability* and *endoscopy*, hence a good testing ground for Langlands program for coverings (Weissman).

Notations. Let $\Pi(H)$ be the admissible dual of a reductive H. Write $\Phi(H)$, etc. for the set of L-parameters. The discrete L^2 automorphic spectrum is denoted by $\mathcal{A}_{\mathrm{disc}}(H)$.

We want to understand $\Pi_-(\tilde{G})$, the set of isomorphism classes of *genuine representations* of $\tilde{G}$, on which $z \in \mu \subset \mathbb{C}^\times$ acts as $z \cdot \mathrm{id}$.

Guiding principle

$\tilde{G}$ behaves as $H := \mathrm{SO}(2n+1)$ (split). The θ-correspondence for pairs $(\mathrm{Sp}(2n), \mathrm{O}(V, q))$ with $\dim V = 2n+1$ provides strong evidence.

In particular, we take $\tilde{G}^\vee = \mathrm{Sp}(2n, \mathbb{C}) = H^\vee$ with trivial Γ_F-action. Whence the notion of L-parameters and A-parameters.

Endoscopic classification for $H := \mathrm{SO}(2n+1)$

The most complete results are provided by trace formula.

$$\Pi_{\mathrm{temp}}(H) = \bigsqcup_{\phi} \Pi_{\phi}, \quad \phi \in \Phi_{\mathrm{bdd}}(H).$$

The internal structure of each packet Π_ϕ is controlled by

$$\begin{aligned} S_\phi^H &:= Z_{H^\vee}(\mathrm{Im}(\phi)), \\ \bar{S}_\phi^H &:= S_\phi^H / Z(H^\vee), \\ \bar{\mathcal{S}}_\phi^H &:= \pi_0(\bar{S}_\pi^H, 1) \quad \text{(finite abelian)}. \end{aligned}$$

Elliptic endoscopic data are in bijection with (n', n'') with $n' + n'' = n$, identifying (n', n'') and (n'', n'); the corresponding endoscopic group is $H^! := \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$.

We have a “Whittaker-normalized” injection

$$\Pi_\phi \hookrightarrow \mathrm{Irr}(\bar{\mathcal{S}}^H_\phi), \text{ bijective for non-archimedean } F.$$
$$\pi \mapsto \langle \cdot, \pi \rangle.$$

Given $\phi \in \Phi_{\mathrm{bdd}}(H)$ and $s \in \bar{S}^H_{\phi,\mathrm{ss}}$, we have

- an endoscopic datum determined by s, the endoscopic group $H^!$ satisfies $\check{H}^! = Z_{H^\vee}(s)$;
- a factorization

$$\phi = WD_F \xrightarrow{\phi^!} \check{H}^! \hookrightarrow \check{H}, \quad \phi^! \in \Phi_{\mathrm{bdd}}(H^!);$$

- the Whittaker-normalized transfer $f \rightsquigarrow f^!$ of test functions.

Define

$$f^!(\phi, s) = f^!(\phi^!) = \sum_{\sigma \in \Pi_{\phi^!}} f^!(\sigma), \quad f^!(\sigma) := \mathrm{tr}(\sigma(f^!)).$$

The character relation is: $\forall(\phi, s)$ as above,

$$f^!(\phi, s) = \sum_{\pi \in \Pi_\phi} \langle s, \pi \rangle f(\pi), \quad f(\pi) = \mathrm{tr}(\pi(f)).$$

In particular, $f^!(\phi, s)$ depends only on the image of s in $\bar{\mathcal{S}}_\phi^H$.

1. For general A-parameters ψ, we will have to multiply s by some s_ψ in $\langle \cdots \rangle$.
2. *Vogan packets*: define $\mathcal{S}_\phi^H := \pi_0(S_\phi^H, 1)$ (abelian 2-group), then

$$\begin{array}{ccccc} \pi & \in & \Pi_\phi & \subset & \bigsqcup_{(V,q)/\simeq} \Pi(\mathrm{SO}(V,q)) \\ \big\downarrow & & {\scriptstyle 1:1}\big\updownarrow & & \\ \langle \cdot, \pi \rangle & \in & \mathrm{Irr}(\mathcal{S}_\phi^H) & & \end{array}$$

where (V, q) ranges over the $(2n+1)$-dimensional quadratic forms with discriminant 1.

- Globally, given a global tempered ("generic") discrete parameter ϕ and $\pi_v \in \Pi_{\phi_v}$ for all v, we have

$$\operatorname{mult}\left(\bigotimes_v' \pi_v : \mathcal{A}_{\text{disc}}(H) \right) = \begin{cases} 1, & \prod_v \langle \cdot, \pi_v \rangle|_{\bar{\mathcal{S}}_\phi^H} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- For A-parameters ψ: require $\prod_v \langle \cdot, \pi_v \rangle|_{\bar{\mathcal{S}}_\phi^H} = \varepsilon_\psi$ instead, where ε_ψ is defined in terms of symplectic root numbers.
- These results are established by a long induction using the full force of stable twisted trace formulas in [Arthur].

Back to $\tilde{G}$

There is a theory of endoscopy for $\tilde{G}$ (Adams, Renard, ..., L.) with transfer $f \rightsquigarrow f^!$ and fundamental lemma for unit.

- Elliptic endoscopic data are in bijection with $\{(n', n'') : n' + n'' = n\}$, the corresponding endoscopic group being $G^! := \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$.
- Principle: similar to H, but should disregard the symmetries from $Z(\tilde{G}^\vee) = \{\pm 1\}$.
- Consequence 1: (n', n'') and (n'', n') play different roles in endoscopy.
- Consequence 2: packets of $\tilde{G}$ (say tempered) "look like" Vogan packets of H.

As before, define S_ϕ, $\mathcal{S}_\phi$ for parameters $\phi \in \Phi(\tilde{G}) = \Phi(H)$.

Non-endoscopic classification of genuine representations: local case

Due to Waldspurger ($n = 1$), Adams-Barbasch (F archimedean) and Gan-Savin (F non-archimedean). Fix ψ_F, etc.

$$\Pi_-(\tilde{G}) \xleftrightarrow{1:1} \bigsqcup_{\substack{(V,q)/\simeq \\ \dim=2n+1,\ \mathrm{disc}=1}} \Pi(\mathrm{SO}(V,q))$$

$$0 \neq \theta_{\psi_F}(\tilde{\pi}^!) \longleftarrow\!\!\dashv \pi^!$$

for a unique extension $\tilde{\pi}^!$ of $\pi^!$ to $\mathrm{O}(V,q)$. This is known to preserve tempered representations, discrete series, etc. Classification of RHS into Vogan packets leads to a substitute of LLC

$$\Pi_-(\tilde{G}) = \bigsqcup_{\phi \in \Phi_{\mathrm{bdd}}(\tilde{G})} \Pi_\phi^{\tilde{G}}.$$

If $\pi = \theta_{\psi_F}(\tilde{\pi}^!) \in \Pi_{\text{temp},-}(\tilde{G})$ as above, set $\langle \cdot, \pi \rangle_\Theta := \langle \cdot, \pi^! \rangle$, a character of $\mathcal{S}_\phi$. Then $\pi \mapsto \langle \cdot, \pi \rangle_\Theta$ is a bijection $\Pi_\phi^{\tilde{G}} \to \mathrm{Irr}(\mathcal{S}_\phi)$.

Tempered parameters for $\tilde{G}$ or H

Write (ϕ, V_ϕ) as a sum of irreducible unitarizable representations of WD_F: $\phi = (\sum_{i \in I_\phi^+} \boxplus \sum_{i \in I_\phi^-}) \ell_i \phi_i \boxplus \sum_{i \in J_\phi} \ell_i (\phi_i \boxplus \check{\phi}_i)$, where

- I_ϕ^+: symplectic ϕ_i;
- I_ϕ^-: orthogonal ϕ_i, with ℓ_i even;
- J_ϕ: non-selfdual ϕ_i.

Hence $S_\phi = \prod_{i \in I_\phi^+} O(\ell_i, \mathbb{C}) \times \prod_{i \in I_\phi^-} \mathrm{Sp}(\ell_i, \mathbb{C}) \times \prod_{i \in J_\phi} \mathrm{GL}(n_i, \mathbb{C})$, and $\mathcal{S}_\phi = \{\pm 1\}^{I_\phi^+}$. Same for global parameters.
Discrete parameters: $I_\phi^- = J_\phi = \emptyset$ and $\forall \ell_i \leq 1$.

Global case: Gan--Ichino formula

Choose $\psi_F : \mathbb{A}_F/F \to \mathbb{C}^\times$. For a global tempered parameter ϕ and $s \in S_{\phi,\mathrm{ss}}$, define

$$\epsilon\left(\frac{1}{2}, V_\phi^{s=-1}, \psi_F\right) = \prod_v \cdots, \text{ say by doubling method.}$$

Only the symplectic summands contribute, by [Lapid] or [Arthur].

- The global $\varepsilon(\frac{1}{2}, \cdots)$ is independent of ψ_F and defines a character $\mathcal{S}_\phi \to \{\pm 1\}$.
- Gan–Ichino: in the discrete genuine automorphic spectrum

$$\mathrm{mult}\left(\bigotimes_v{}' \pi_v : \mathcal{A}_{\text{disc,-}}(\tilde{G})\right) = \begin{cases} 1, & \prod_v \langle s, \pi_v \rangle_\Theta = \epsilon\left(\frac{1}{2}, V_\phi^{s=-1}, \psi_F\right) \\ 0, & \text{otherwise.} \end{cases}$$

Note. Can also formulate the case for general A-parameters.

Towards the local endoscopic classification

Desiderata. Character relations for $\tilde{G}$ via endoscopic transfer, and description of the coefficients.
Keep the notations for the case $H = \mathrm{SO}(2n+1)$. Known:

$$\left[f^!(\phi^!) =: f^!(\phi, s)\right] = \sum_{\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})} \Delta(\phi^!, \pi) f(\pi), \quad s \in S_{\phi,\mathrm{ss}}.$$

- f: anti-genuine C_c^∞ function on $\tilde{G}$ that probes the genuine spectrum, and $f \mapsto f(\pi)$ is the character;
- $f \rightsquigarrow f^!$ is the transfer for the endoscopic datum determined by s, it factorizes ϕ into $\phi^!$.
- RHS: virtual character with unknown coefficients. Assuming that π determines ϕ (LLC), we write

$$\Delta(\phi^!, \pi) = \langle s, \pi \rangle.$$

Minimal requirements.

1. When $s = 1$ and $\phi \in \Phi_{\mathrm{bdd}}(\tilde{G})$, should have $\langle s, \pi \rangle \in \{0, 1\}$, i.e. $f^!(\phi, 1)$ = "stable character" on $\tilde{G}$.
2. $\langle s, \pi \rangle$ depends only on the conjugacy class of s in $S_{\phi,\mathrm{ss}}$.

Key property

Given an elliptic endoscopic datum (n', n'') for $\tilde{G}$, denote by $\mathcal{T}_{(n',n'')}$ the dual of transfer. The following commutes.

$$\begin{array}{ccc} \{\text{st. dist. on } \mathrm{SO}_{2n'+1} \times \mathrm{SO}_{2n''+1}\} & \xrightarrow{\mathcal{T}_{(n',n'')}} & \{\text{genuine inv. dist. on } \tilde{G}\} \\ \text{swap} \downarrow \simeq & & \simeq \downarrow \text{translate by } -1 \\ \{\text{st. dist. on } \mathrm{SO}_{2n''+1} \times \mathrm{SO}_{2n'+1}\} & \xrightarrow[\mathcal{T}_{(n'',n')}]{} & \{\text{genuine inv. dist. on } \tilde{G}\} \end{array}$$

Consequence: Assuming $s^2 = 1$, we have $\langle -s, \pi \rangle = \omega_\pi(-1)\langle s, \pi \rangle$. Call $\omega_\pi(-1)$ the *central sign* of π.

Case study: $n = 1 \Rightarrow$ already have LLC for $\tilde{G}$ (Waldspurger, Adams, Schultz). Let $\phi \in \Phi_{\mathrm{bdd}}(\tilde{G})$ and $\pi \in \Pi_\phi$. Results of Gan–Savin et al. imply:

- If π comes from $\mathrm{SO}(V, q)$ by θ-lifting, $\dim V = 3$ with discriminant 1, then $\langle -1, \pi \rangle_\Theta \in \{\pm 1\}$ equals the Hasse invariant of (V, q).
- The observation above says $\langle -1, \pi \rangle = \omega_\pi(-1)$, which turns out to be $\langle -1, \pi \rangle_\Theta \varepsilon\left(\frac{1}{2}, \phi, \psi_F\right)$.

Taking $\phi = \phi_0 \boxplus \check{\phi}_0$ with $\phi_0 \neq \check{\phi}_0$, we get $S_\phi = \mathrm{GL}(1, \mathbb{C})$, $\langle \cdot, \pi \rangle_\Theta = 1$ and $\langle -1, \pi \rangle = \phi_0 \circ \mathrm{rec}_F(-1)$ is not always trivial. A closer look reveals that

1. $\langle \cdot, \pi \rangle$ does not factor through $\mathcal{S}_\phi$;
2. it is even not a homomorphism in general;
3. $\langle s, \pi \rangle = \varepsilon\left(\frac{1}{2}, V_\phi^{s=-1}, \psi_F\right) \langle s, \pi \rangle_\Theta$ for all $s \in S_{\phi, \mathrm{ss}}$.

Global reasons: for $\phi \in \Phi_2(\tilde{G})$, general arguments from Arthur suggest

$$\mathrm{mult}\left(\bigotimes_v{}' \pi_v : \mathcal{A}_{\mathrm{disc},-}(\tilde{G}) \right) = |\mathcal{S}_\phi|^{-1} \sum_{s \in \mathcal{S}_\phi} \prod_v \langle s, \pi_v \rangle.$$

The local observations ($n = 1$ at least) + Gan–Ichino formula give some support for the following.

Conjecture

We have LLC $\Pi_{\mathrm{temp},-}(\tilde{G}) = \bigsqcup_{\phi \in \Phi_{\mathrm{bdd}}(\tilde{G})} \Pi_\phi$ with character relations. Assigning to $\pi \in \Pi_\phi$ the coefficients $\langle \cdot, \pi \rangle : S_{\phi,\mathrm{ss}} \to \mathbb{C}^\times$ gives

$$\Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi)\xi_\phi, \quad \xi_\phi(s) := \varepsilon\left(\frac{1}{2}, V_\phi^{s=-1}, \psi_F \right).$$

Note that

1. ξ_ϕ is not a character of $\mathcal{S}_\phi$, although their global product is;
2. for $\phi \in \Phi_2(\tilde{G})$, we can show $\xi_\phi \in \mathrm{Irr}(\mathcal{S}_\phi)$;
3. can formulate a version for A-packets.

Refinement of the conjecture

We expect $\langle \cdot, \pi \rangle = \langle \cdot, \pi \rangle_\Theta \xi_\phi$ for all $\phi \in \Phi_{\mathrm{bdd}}(\tilde{G})$ and $\pi \in \Pi_\phi$.

- These will completely reconcile the endoscopic and the θ-lifting descriptions for $\Pi_{\mathrm{temp},-}(\tilde{G})$.
- The case of epipelagic L-packets for $F \supset \mathbb{Q}_p$, $p \gg 0$ seems accessible by purely local arguments [Kaletha 2015].

Strategy

A template: Arthur's *Endoscopic classification of representations* and *Unipotent automorphic representations*. The proof is necessarily local-global.

The case for $\tilde{G}$ ought to be easier, since

- stable multiplicity formula for endoscopic groups are available,
- candidates for global parameters are already well-defined,
- one can resort to Gan-Ichino in local-global arguments if necessary.

An important ingredient in Arthur's "Standard Model": local and global intertwining relations.

Take F local, fix ψ_F and F-pinnings for all groups. Consider a standard parabolic $\tilde{P} = \tilde{M}U \subsetneq \tilde{G}$ and their dual $\tilde{P}^\vee, \tilde{M}^\vee \subset \tilde{G}^\vee$. Assume LLC, etc. for $\tilde{M}$.

- Take $\phi_M \in \Phi_{2,\mathrm{bdd}}(\tilde{M})$ (for simplicity), $\pi_M \in \Pi_{\phi_M}$.
- Let $w \in W_\phi \subset W(M) = W(\tilde{M}^\vee)$.

Let $\phi_M \mapsto \phi \in \Phi_{\mathrm{bdd}}(\tilde{G})$. Consider normalized intertwining operators of the form

$$R_P(w, \pi_M, \phi) = \pi(w)\ell(w, \pi_M, \phi)R_{wPw^{-1}|P}(\pi_M, \phi)$$
$$I_{\tilde{P}}(\pi_M) \to I_{w\tilde{P}w^{-1}}(\pi_M) \to I_{\tilde{P}}(w\pi_M) \to I_{\tilde{P}}(\pi_M).$$

- First, must lift $w \in W^G(M)$ to an element of $\tilde{G}$. Just use the Springer section with due care on long simple roots, eg. we want $w_{\mathrm{long}}^2 = -1$ in $\widetilde{\mathrm{Sp}}(2, F)$.
- Group-theoretic difficulties of this step are addressed in [Brylinski-Deligne, 2001].

- $R_{wPw^{-1}|P}(\pi_M, \phi)$ is the normalized intertwining operator, we can use the normalizing factors from $\mathrm{SO}(2n+1)$.
- $\ell(w, \pi_M, \phi)$ is $\phi \mapsto \phi(w^{-1}\cdot)$ times some correction factor, again using that from $\mathrm{SO}(2n+1)$-case.
- Choose $\pi(w) \in \mathrm{Isom}_{\tilde{M}}(w\pi_M, \pi_M)$: it affects only the GL-slots of π_M.

The resulting operator is neither multiplicative in w nor Whittaker-normalized! Denote by $\gamma(w, \phi) \in \mathbb{C}^\times$ the expected "effect of $R_P(w, \pi_M, \phi)$", say on Whittaker functionals. See eg.

- Bump-Friedberg-Hoffstein (1991),
- Szpruch (2013).

Again, local root numbers appear here (general phenomena).

Take any $u \in N_\phi := N_{S_\phi}(A_{\tilde{M}^\vee})$, which maps to $w \in W_\phi$ under the natural $N_\phi \to W_\phi$. There is a natural splitting $\pi_0(N_\phi) = \mathcal{S}_{\phi_M} \times W_\phi$. Write $u = u_0 \cdot u^\flat$ where $u^\flat \in S_{\phi_M}$ and u have the same image in $\mathcal{S}_{\phi_M}$.

Special case of local intertwining relation

Conjecturally $f^!(\phi, u) = f(\phi, u)$ (see below).

$$f(\phi, u) := c(u^\flat, \phi) \sum_{\pi_M \in \Pi_{\phi_M}} \langle \pi_M, u^\flat \rangle \mathrm{tr}\left(R_P(w, \pi_M, \phi) I_{\tilde{P}}(w, f) \right)$$

$$c(u^\flat, \phi) := \varepsilon\left(\frac{1}{2}, V_\phi^{u=-1}, \psi_F\right) \varepsilon\left(\frac{1}{2}, V_{\phi^\flat}^{u^\flat=-1}, \psi_F\right)^{-1} \gamma(w, \phi)^{-1}.$$

This will give the $\langle \cdot, \pi \rangle$ for tempered non-L^2 packets, and much more (eg. information on R-groups).
See [Arthur, Chapter 2] for full explanation.

The local intertwining relation is subject to global constraints.

1. $\prod_v c(u, \phi)_v = 1$ for global parameters.
2. Generalizes to A-packets.
3. Satisfy analogues of the *sign lemmas* in [Arthur, Chapter 4].
4. Ultimately, we want to feed them into the *Standard Model* of *loc. cit.*, and deduce all the theorems inductively.

We will need

- the fundamental lemma for the spherical Hecke algebra of $\tilde{G}$ (ongoing thesis of Caihua Luo supervised by Gan);
- probably also the full stable trace formula.