On a lifting of Ikeda–Yamana

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SUSTech

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Genesis: Ikeda (2001)

$$f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z})) \qquad \text{nor. eigenform, } k \in \mathbb{Z}_{\geqslant 1}$$

$$\downarrow$$

$$\mathcal{F} \in S^{(n)}_{k+\frac{n}{2}}(\mathrm{Sp}_{2n}(\mathbb{Z})) \qquad \text{nor. eigenform, } n \text{ even, } k \equiv \tfrac{n}{2} \bmod 2$$

such that $L(s, \mathcal{F}) = \zeta(s) \prod_{i=1}^{n} L(s+k+n-i, f)$.

Here Sp_{2n} is the symplectic group of rank n, and $S^{(n)}_{k+\frac{n}{2}}$ is the space of Siegel cusp forms of weight $k + \frac{n}{2}$ and level Γ (to be reviewed).

- Conjectured by Duke–Imamoglu, generalizing Saito–Kurokawa lifting.
- Done explicitly by writing down an Fourier–Jacobi expansion for $\mathcal{F}$.
- Many applications in number theory.

Hilbert–Siegel modular forms

- F: totally real number field, $[F:\mathbb{Q}] = d$, $n \in \mathbb{Z}_{\geqslant 1}$. Fix an additive character ψ of $F\backslash(\mathbb{A} := \mathbb{A}_F)$.
- $\mathcal{H}_n$: the Siegel upper half-plane of degree n, on which $\mathrm{Sp}_{2n}(\mathbb{R})$ acts.
- $\mathrm{Mp}_{2n}(\mathbb{R}) \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbb{R})$: non-trivial twofold covering of Lie groups, not algebraic (called *metaplectic covering*).
- **Weight** d-tuple $\ell = (\ell_v)_{v|\infty}$ where $\ell_v \in \frac{1}{2}\mathbb{Z}$, with $2\ell_v \equiv 2\ell_w \pmod 2$ for all $v, w \mid \infty$.
- **Automorphy factor** $J_\ell(\tilde{g}, Z) = \prod_{v|\infty} j(\tilde{g}_v, Z_v)^{2\ell_v}$ where $\tilde{g} = (\tilde{g}_v)_{v|\infty}$, $Z = (Z_v)_{v|\infty}$, with

$$j : \mathrm{Mp}_{2n}(\mathbb{R}) \times \mathcal{H}_n \to \mathbb{C}^\times, \quad j(\tilde{g}_v, Z_v)^2 = \det(C_v Z_v + D_v)$$

$$\text{where} \quad \tilde{g}_v \mapsto \begin{pmatrix} * & * \\ C_v & D_v \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}).$$

- When $\forall \ell_v \in \mathbb{Z}$, the $J_\ell(\cdot, \mathcal{Z})$ descends to $\mathrm{Sp}_{2n}(\mathbb{A}_\infty)$.
- Otherwise $J_\ell(\cdot, \mathcal{Z})$ can be defined on not-too-big congruence subgroups Γ.

Hilbert–Siegel modular forms of weight ℓ and level Γ

These are holomorphic functions $\mathcal{F} : \prod_{v|\infty} \mathcal{H}_n \to \mathbb{C}$ such that

$$\mathcal{F}(\gamma\mathcal{Z}) = J_\ell(\gamma, \mathcal{Z})\mathcal{F}(\mathcal{Z}), \quad \forall \gamma \in \Gamma,\ \mathcal{Z} \in \prod_{v|\infty} \mathcal{H}_n,$$

plus conditions at cusps if $n = 1$ and $F = \mathbb{Q}$.

We get the spaces of modular and cusp forms $M_\ell^{(n)}(\Gamma) \supset S_\ell^{(n)}(\Gamma)$.

Adelic interpretation:

- If $\forall \ell_v \in \mathbb{Z}$, these are (certain) automorphic forms on $\mathrm{Sp}_{2n}(\mathbb{A})$.
- If $\forall \ell_v \in \frac{1}{2} + \mathbb{Z}$, we need to pass to a twofold covering.

Below, F can be any number field, $\boldsymbol{\mu}_2 := \mu_2(\mathbb{C})$.

- At each place v, we have a topological central extension

$$1 \to \boldsymbol{\mu}_2 \to \underbrace{\mathrm{Mp}_{2n}(F_v)}_{\text{non alg. unless } F=\mathbb{C}} \to \mathrm{Sp}_{2n}(F_v) \to 1.$$

- Let $\mathrm{Mp}_{2n}(\mathbb{A}) := \prod_v' \mathrm{Mp}_{2n}(F_v)/\text{junk}$, then

$$1 \to \boldsymbol{\mu}_2 \to \mathrm{Mp}_{2n}(\mathbb{A}) \to \mathrm{Sp}_{2n}(\mathbb{A}) \to 1,$$

  which splits canonically over $\mathrm{Sp}_{2n}(F)$.

- They are called *metaplectic coverings* or *metaplectic groups*.
- We study *genuine* representations and genuine automorphic forms on them. "Genuine" means: μ_2 acts tautologically.
- Genuine automorphic forms/representations provide a natural set-up for studying Hilbert–Siegel modular forms of $\frac{1}{2} + \mathbb{Z}$ weights (Weil, Shimura, Waldspurger...)

Some examples

- Classically, Θ-series of unimodular lattices can give rise to modular forms of half-integral weight.
- Θ-correspondence involves metaplectic groups.

Revisiting Ikeda lifting

To $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ are attached:

1. a cuspidal automorphic representation $\pi = \bigotimes_v' \pi_v$ of $\mathrm{PGL}_2(\mathbb{A})$;
2. an L-parameter $\phi_0 : \mathcal{L}_{\mathbb{Q}} \to \mathrm{PGL}_2^\vee = \mathrm{SL}_2(\mathbb{C})$;
3. the composite

$$\psi : \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\phi_0 \boxtimes r(n)} \mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{SO}_{2n+1}(\mathbb{C}) = \mathrm{Sp}_{2n}^\vee$$

where $r(n)$ = the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$, symplectic for even n, orthogonal for odd n.

Arthur's multiplicity formula explains the lifting $\mathcal{F}$: it is parameterized by the *Arthur parameter* ψ for Sp_{2n}!

Remark

The automorphic Langlands group $\mathcal{L}_{\mathbb{Q}}$ is hypothetical, however the statements still make sense (see Arthur's book).

Ikeda–Yamana (2020)

They generalize [Ikeda] to Hilbert modular forms for PGL_2 over totally real F, general level, and the n can be odd.

$$\begin{array}{ll} f: & \text{Hilbert nor. eigenform, weight} = 2k,\ \forall k_v \in \mathbb{Z}_{\geqslant 1} \\ \downarrow & \\ \mathcal{F}: & \text{Hilbert–Siegel cusp form } \in S^{(n)}_{k+\frac{n}{2}}. \end{array}$$

Here f generates $\pi = \bigotimes'_v \pi_v$, and they impose

- some "parity conditions" on π_{fini},
- $k_v > \frac{n}{2}$ for all $v \mid \infty$.

When n is even (resp. odd), $\mathcal{F}$ lives on $\mathrm{Sp}_{2n}(\mathbb{A})$ (resp. $\mathrm{Mp}_{2n}(\mathbb{A})$).

- The proof is a beautiful blend of explicit Fourier–Jacobi expansion + representation theory.
- In terms of Arthur parameters (say for **odd** n): π has L-parameter $\phi_\circ$, and we obtain

$$\psi : \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}(2, \mathbb{C}) \xrightarrow{\phi_\circ \boxtimes r(n)} \mathrm{Sp}_{2n}(\mathbb{C}) =: \mathrm{Mp}_{2n}^\vee .$$

  The last $=:$ is an instance of *Langlands' program for covering groups*, in the most accessible (yet nontrivial) case of Mp_{2n}. See Gan's ICM talk or [Gan–Gao–Weissman] for details.
- In particular, the Ikeda–Yamana lifting for odd n should be "explained" by **Arthur's multiplicity formula for** Mp_{2n}.

The lifting is best phrased in terms of…

Yamana's conjecture

Reference: Shunsuke Yamana, *The CAP representations indexed by Hilbert cusp forms*. arXiv:1609.07879

Let n be odd, F: totally real, $\psi = \prod_v \psi_v : F\backslash\mathbb{A} \to \mathbb{C}^\times$ satisfies $\psi_v(x) = e^{2\pi\sqrt{-1}\cdot x}$ for all $v \mid \infty$ and $x \in \mathbb{R}$.

- Let π be generated by a Hilbert cusp form for PGL_2 over F of weight $2k$, where $k = (k_v)_{v|\infty}$ with $k_v \in \frac{1}{2} + \mathbb{Z}$ for all $v \mid \infty$.
- To π is attached $\phi_\circ : \mathcal{L}_F \to \mathrm{SL}_2(\mathbb{C}) = \mathrm{Mp}_2^\vee$ (use Arthur's makeshift parameters to get rid of $\mathcal{L}_F$).
- From $\phi_\circ$ we obtain Waldspurger's packet $\{\pi_v^+, \pi_v^-\}$ of genuine irreducible representations of $\mathrm{Mp}_2(F_v)$, at each place v.

We will define genuine irreducible representations Π_v^+, Π_v^- of $\mathrm{Mp}_{2n}(F_v)$ from $\phi_\circ$, at each place v.

- **Non-Archimedean case** Define $\Pi_v^{\pm}$ as the Langlands quotient

$$\underbrace{|\det|_v^{\frac{n-1}{2}}\pi_v \boxtimes \cdots \boxtimes |\det|_v \pi_v}_{\frac{n-1}{2}\text{ copies of }\mathrm{GL}_2} \boxtimes \pi_v^{\pm} \twoheadrightarrow \Pi_v^{\pm}.$$

They are expected to appear in the *Arthur packet* Π_{ψ_v} (a multi-set of unitary irreducible genuine representations) with multiplicity one.

- **Real case** The metaplectic covering restricts to the unique non-split twofold covering $\tilde{U}(n) \to U(n)$. Consider $\ell \in \frac{1}{2} + \mathbb{Z}$ with $\ell > 0$. Define
  - $D_\ell^{(n)}$: the *lowest weight module* of $\mathrm{Mp}_{2n}(\mathbb{R})$ with lowest $\tilde{U}(n)$-type $\det^\ell$ ($\implies$ holomorphic);
  - $\overline{D}_\ell^{(n)}$: the *highest weight module* of $\mathrm{Mp}_{2n}(\mathbb{R})$ with highest $\tilde{U}(n)$-type $\det^{-\ell}$ ($\implies$ anti-holomorphic).

$$\Pi_v^{(-1)^{\frac{n-1}{2}}} := D_{k_v+\frac{n}{2}}^{(n)}, \quad \Pi_v^{(-1)^{\frac{n+1}{2}}} := \overline{D}_{k_v+\frac{n}{2}}^{(n)}.$$

Consider $(\epsilon_v)_{v:\text{place}}$ with $\epsilon_v \in \{\pm 1\}$, assume $\epsilon_v = 1$ for almost all v.

Conjecture (Yamana)

The genuine representation $\bigotimes_v' \Pi_v^{\epsilon_v}$ of $\mathrm{Mp}_{2n}(\mathbb{A})$ occurs with multiplicity 1 in the cuspidal automorphic spectrum when $\prod_v \epsilon_v = \epsilon\left(\frac{1}{2}, \pi\right)$, and has multiplicity 0 otherwise.

When $n = 1$, this recovers Waldspurger's celebrated results for $\mathrm{Mp}_2(\mathbb{A})$.

Theorem (essentially in [Ikeda–Yamana])

If π_v is non-supercuspidal for all $v \nmid \infty$, and $k_v > \frac{n}{2}$ for all $v \mid \infty$, then the conjecture holds true.

Main results

Theorem in progress (L.)

The conjecture holds true if the cuspidal automorphic spectrum is replaced by the discrete L^2-automorphic one.

If $k_v > \frac{n}{2}$ for all $v \mid \infty$, then $\Pi_v^{\pm}$ are tempered (in fact L^2), so $L^2_{\mathrm{cusp}} = L^2_{\mathrm{disc}}$ (a well-known result of Wallach).

The main ingredients include:

- Arthur packets for Mp_{2n} (local);
- Arthur's multiplicity formula for Mp_{2n} (global);
- multiplicity-one of $\Pi_v^{\pm}$ in the Arthur packet Π_{ψ_v}, where $\psi := \phi_0 \boxtimes r(n) : \mathcal{L}_F \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$.

They are also works in progress, nearing completion IMAO.

Arthur's conjectures predict (among others)

$$L^2_{\mathrm{disc}}(\mathrm{Sp}_{2n}(F)\backslash \mathrm{Mp}_{2n}(\mathbb{A})) = \widehat{\bigoplus_{\xi}} L^2_{\xi}, \qquad \xi : \text{“discrete” Arthur parameters.}$$

This decomposition is already done by Gan–Ichino (2018).

- If $\bigotimes'_v \Pi_v^{\epsilon_v}$ occurs in L^2_{disc}, it must occur in L^2_{ψ} by consideration of Satake parameters.
- One can then apply Arthur's multiplicity formula for Mp_{2n} (in progress) + multiplicity-one (see below) to conclude. The appearance of $\epsilon\left(\frac{1}{2}, \pi\right)$ here is a *metaplectic feature* here!

When $v \nmid \infty$, multiplicity-one of $\Pi_v^{\pm}$ in Π_{ψ_v} is part of the general properties of Arthur packets, as $\Pi_v^{\pm}$ are defined as Langlands quotients.

$\rightsquigarrow$ Remains to show multiplicity-one for $v \mid \infty$. A **real problem**.

Sketch of the strategy over $\mathbb{R}$ (in progress)

Remark

Assuming that a theory à la Adams–Johnson of (certain) Arthur packets of $\mathrm{Mp}_{2n}(\mathbb{R})$ exists, then multiplicity-one will follow.

Instead, we try an *ad hoc* approach as follows.

- Step 1 ($k > \frac{n}{2}$): Globalize to $\mathbb{Q}$, combine the multiplicity formula with the results of Ikeda–Yamana to get multiplicity one.
- Step 2 ($k \in \mathbb{Z}_{\geqslant 1}$): Use Zuckerman's *translation functor*.
  1. Translation commutes with transfer — imitate the arguments of Moeglin–Renard.
  2. Translation preserves highest/lowest weight modules (look at Verma modules).
  3. No loss of information in translation: use the equi-singularity of the infinitesimal characters in question.

  In this way, we hope to deduce multiplicity-one from Step 1.

Main references

1. James Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*. AMS Coll. Vol. 61 (2013).
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3. Tamotsui Ikeda and Shunsuke Yamana, *On the lifting of Hilbert cusp forms to Hilbert-Siegel cusp forms*. ASENS tome 53, fascicule 5 (2020).
4. Wee Teck Gan, Fan Gao and Martin Weissman, *L-groups and the Langlands program for covering groups*, Astérisque No. 398 (2018).
5. Shunsuke Yamana, *The CAP representations indexed by Hilbert cusp forms*. arXiv:1609.07879

Thanks for your attention

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