Transcending Endoscopy

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Motivations: the Arthur-Selberg Trace Formula

As you have known, the Arthur-Selberg invariant trace formula is one of the main tools to study automorphic representations of reductive groups. It reads $I(f) = \sum \text{geometric terms} = \sum \text{spectral terms}$.

- Geometric side: weighted orbital integrals (in fact, it mixes some spectral data...).
- Spectral side: weighted characters.
- Basic idea: relate the spectral data of different groups by comparing conjugacy classes.
- Difficulty: usually one can only compare stable conjugacy classes, and stable conjugacy $\neq$ conjugacy in general.

Stabilization: rewrite the trace formula in terms of stable distributions on so-called endoscopic groups. The first step is to stabilize the elliptic regular terms on the geometric side; they consist of orbital integrals of elliptic regular orbits.

The technical bottleneck is the Langlands-Shelstad transfer conjecture and the so-called fundamental lemma. They are recently proven by Ngô, based on the works of Waldspurger, Kottwitz, Langlands, Shelstad and many others...

The metaplectic groups

Let F be a local field. Fix a nontrivial character $\psi : F \to \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$.
We are concerned with some nonalgebraic covering $p : \widetilde{\mathrm{Sp}}(2n, F) \to \mathrm{Sp}(2n, F)$. Why bother about such nonalgebraic things?

- It gives an adélic interpretation of Siegel modular forms of half-integral weight, eg. the theta series.
- It carries the Weil representation ω_ψ. This representation underlies the Howe correspondence, also known as "transcendental classical invariant theory". Thus the title of my talk.

Desiderata

1. Establish the invariant trace formula for $\widetilde{\mathrm{Sp}}(2n, F)$.
2. For appropriate test functions (will be defind later), formulate a variant of the Langlands-Transfer conjecture. It turns out that the elliptic endoscopic groups are $\mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$ (the split odd SO) where $n' + n'' = n$.
3. In particular, $\mathrm{SO}(2n+1)$ is in some sense the "quasisplit inner form" of $\widetilde{\mathrm{Sp}}(2n, F)$. This is closely related to the Howe correspondence for the dual pair $(O(2n+1), \mathrm{Sp}(2n))$.
4. In the real case, Adams studied the case $n = n'$ by using character theory. Renard then established the transfer of orbital integrals for general (n', n'').
5. For p-adic fields, Schultz established the character identities in the case $n = 1, n'' = 0$.

Representations of the Heisenberg group

Let $(W, \langle | \rangle)$ be a symplectic space over F. Let $H(W) = W \times F$ be the associated Heisenberg group, the group law being

$$(w, t) \cdot (w', t') = \left(w + w', t + t' + \frac{\langle w | w' \rangle}{2} \right).$$

Then the center of $H(W)$ is just $\{0\} \times F$.

Stone-von Neumann Theorem

$H(W)$ admits a unique irreducible unitary representation of central character ψ. Denote it by (ρ_ψ, S_ψ).
When F is nonarchimedean, one can replace "unitary" by "smooth". ρ_ψ is necessarily admissible.

$\mathrm{Sp}(W)$ acts on $H(W)$ by $g \cdot (w, t) = (gw, t)$. For $g \in \mathrm{Sp}(W)$, put $\rho_\psi^g : h \mapsto \rho_\psi(g \cdot h)$. By the theorem above, ρ_ψ^g and ρ_ψ are intertwined by some operator $M[g]$, unique up to $\mathbb{C}^\times$.

Construction of the metaplectic group

The "big" metaplectic group $\overline{\mathrm{Sp}}_\psi(W)$ is just the subgroup of those $(g, M[g]) \in \mathrm{Sp}(W) \times \mathrm{Aut}(S_\psi)$, where $\mathrm{Aut}(S_\psi)$ is equipped with the strong operator topology. Then $\overline{\mathrm{Sp}}_\psi(W)$ is a locally compact group. The projection $p : \overline{\mathrm{Sp}}_\psi(W) \to \mathrm{Sp}(W)$ gives an extension of $\mathrm{Sp}(W)$ by $\mathbb{C}^\times$. The Weil representation ω_ψ is the other projection $(g, M[g]) \mapsto M[g]$. It decomposes into two nonisomorphic irreducible pieces: $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$.

Fact

The covering p is trivial iff $F \simeq \mathbb{C}$. It can be reduced to an extension by μ_2; denote it by $\widetilde{\mathrm{Sp}}^{(2)}(W)$.

The genuine objects: the functions (resp. representations, etc.) on $\widetilde{\mathrm{Sp}}^{(2)}(W)$ such that $f(zx) = zf(x)$ for all $z \in \mu_2$.
As the name suggests, they are the genuinely interesting objects on the metaplectic group.

On the choice of coverings

For m an even integer, set $\widetilde{\mathrm{Sp}}^{(m)}(W) := \widetilde{\mathrm{Sp}}^{(2)}(W) \times_{\mu_2} \mu_m$ (fibered coproduct). It is an extension of $\mathrm{Sp}(W)$ by μ_m.

Genuine: $f(zx) = zf(x)$, **anti-genuine**: $f(zx) = z^{-1}f(x)$. They coincide if $m = 2$! The adjective genuine (resp. anti-genuine) also applies to distributions, representations, Iwahori-Hecke algebras and spherical Hecke algebras (this makes sense: see the next slide).

In general, one can pass from genuine to anti-genuine by multiplying some character of $\widetilde{\mathrm{Sp}}^{(m)}(W)$. This induces an isomorphism between genuine and anti-genuine Iwahori-Hecke algebras and spherical Hecke algebras. In short, there is little difference......

The Eightfold Way

Our main results will be independent of the choice of m. I claim that $m = 8$ is the best choice, since it makes the metaplectic extension split over Siegel parabolics. Set $\widetilde{\mathrm{Sp}}(W) := \widetilde{\mathrm{Sp}}^{(8)}(W)$ hereafter.

Why 8? It is ultimately due to the fact that the Weil constant $\gamma_\psi : W(F) \to \mathbb{C}^\times$ has image in μ_8, where $W(F)$ is the Witt group of quadratic forms over F.

Splitting over hyperspecial subgroups

Suppose that F is a finite extension of $\mathbb{Q}_p$, $p > 2$. Let $\mathfrak{o}_F$ be the ring of integers and $\mathfrak{p}_F$ the maximal ideal.
If $K \subset \mathrm{Sp}(W)$ is a hyperspecial subgroup, then $p : \widetilde{\mathrm{Sp}}(W) \to \mathrm{Sp}(W)$ splits over K.
In fact, the $\mathrm{Sp}(W)$-orbits of hyperspecial vertices in the Bruhat-Tits building of $\mathrm{Sp}(W)$ are in 1-1 correspondence with lattices $L \subset W$ such that

$$L^* = \mathfrak{p}_F L \text{ or } L^* = L$$

where $L^* := \{w \in W : \forall a \in L, \langle a|w\rangle \in \mathfrak{o}_F\}$. The associated hyperspecial subgroup is $K = \{g \in \mathrm{Sp}(W) : gL = L\}$.
The required splitting over K is obtained by an explicit construction of (ρ_ψ, S_ψ) as $\mathrm{Ind}_{L\times F}^{H(W)}(1 \times \psi)$.
Up to a rescaling of $\langle | \rangle$, it suffices to consider the hyperspecial subgroups associated to a lattice L such that $L^* = L$.

The adélic picture

Suppose that F is a global field. Fix a nontrivial character $\psi : \mathbb{A}/F \to \mathbb{S}^1$ with decomposition $\psi = \prod_v \psi_v$. Use ψ_v to construct $\widetilde{\mathrm{Sp}}(W_v)$ at every place v.

Define the adélic points

$$\widetilde{\mathrm{Sp}}(W, \mathbb{A}) := \frac{\prod_v' \widetilde{\mathrm{Sp}}(W_v)}{\{(z_v)_v \in \bigoplus_v \mu_8 : \prod_v z_v = 1\}}.$$

Then the local projections combine into $p : \widetilde{\mathrm{Sp}}(W, \mathbb{A}) \to \mathrm{Sp}(W, \mathbb{A})$. This gives rise to an eightfold covering $\widetilde{\mathrm{Sp}}(W, \mathbb{A}) \to \mathrm{Sp}(W, \mathbb{A})$.

Weil showed that here is a canonical injection $\mathrm{Sp}(W) \to \widetilde{\mathrm{Sp}}(W, \mathbb{A})$ of discrete image. Thus it makes sense to discuss the automorphic representations, Eisenstein series, etc.. on metaplectic groups.

Basic ideas of endoscopy

Let G be a reductive group over F, local or global. Below is a list of the basic ingredients of endoscopy.

- Endoscopic groups H + other data. For simplicity, assume that ${}^L H \hookrightarrow {}^L G$.
- Correspondence of semisimple stable conjugacy classes between $H(F)$ and $G(F)$. Matching strongly regular classes have isomorphic centralizer.
- A transfer factor Δ.

Now let F be a local field. For $\delta \in G(F)$, let δ_s be its semisimple part. Set

$$D(\delta) := \det(1 - \mathrm{Ad}(\delta_s)|\mathfrak{g}/\mathfrak{g}_{\delta_s}).$$

For $f \in C_c^\infty(G(F))$, define the normalised orbital integrals by

$$J_G(\delta, f) := |D(\delta)|^{1/2} \int_{G_\delta(F)\backslash G(F)} f(x^{-1}\delta x)\, dx.$$

When δ is semisimple strongly regular i.e. δ has connected centralizer, one defines similiarly the normalised stable orbital integral $J_G^{st}(\delta, f) = \sum_{\delta_1 \overset{st}{\sim} \delta} J_G(\delta_1, f)$.

Similarly, for $X \in \mathfrak{g}(F)$, let X_s be its semisimple part. Set

$$D(X) := \det(\mathrm{ad}(X)|\mathfrak{g}/\mathfrak{g}_{X_s}).$$

For $f \in C_c^\infty(\mathfrak{g}(F))$, define the normalised orbital integrals by

$$J_G(X, f) := |D(X)|^{1/2} \int_{G_\delta(F)\backslash G(F)} f(x^{-1}Xx)\, dx.$$

The definition of stable orbital integrals on Lie algebra is similar.

The Transfer Conjecture

Suppose that $\gamma \in H(F)$ is G-regular, i.e. γ is semisimple and it corresponds to some strongly regular semisimple $\delta \in G(F)$. Define the endoscopic orbital integral of f by

$$J_{G,H}(\gamma, f) := \sum_{\delta} \Delta(\gamma, \delta) J_G(\delta, f)$$

where δ ranges over the conjugacy classes in $G(F)$ corresponding to γ. We say $f^H \in C_c^\infty(H(F))$ is a transfer of f if $J_{G,H}(\gamma, f) = J_H^{st}(\gamma, f^H)$ for all strongly G-regular γ, where we use matching Haar measures on $H_\gamma(F) \simeq G_\delta(F)$.

Transfer

For any $f \in C_c^\infty(G(F))$, there exists a transfer f^H.

Fundamental Lemma for Units

Suppose that the endoscopic datum is "unramified" and the residual characteristic p is large enough. If $K \subset G(F)$ is hyperspecial, $f = \mathbb{1}_K$, then one can take f^H to be $\mathbb{1}_{K_H}$ where K_H is any hyperspecial subgroup of $H(F)$. Here we use measures such that K and K_H have mass 1.

Endoscopic groups of $\widetilde{\mathrm{Sp}}(W)$

To find the analogs for $\widetilde{\mathrm{Sp}}(W)$ of all the ingredients above, it is best to start with the unramified case. Let F be local nonarchimedean of residual characteristic $p > 2$. Set $2n := \dim_F W$.

Theorem (Savin)

The genuine Iwahori-Hecke algebra of $\widetilde{\mathrm{Sp}}(W)$ is isomorphic to the Iwahori-Hecke algebra of $\mathrm{SO}(2n+1)$.

The same holds for anti-genuine Iwahori-Hecke algebras.
Thus the dual group of $\widetilde{\mathrm{Sp}}(W)$ should be $\mathrm{Sp}(2n, \mathbb{C})$. Pursuing this analogy one step further, the elliptic endoscopic groups of $\widetilde{\mathrm{Sp}}(W)$ should be $\mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$ with $n' + n'' = n$. Note that they are also the elliptic endoscopic groups of $\mathrm{SO}(2n+1)$.

Point correspondences

This is "almost" the correspondence by eigenvalues. Set $G := \mathrm{Sp}(W)$, $\tilde{G} := \widetilde{\mathrm{Sp}}(W)$, $H := \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$.
Let $\gamma = (\gamma', \gamma'') \in H(F)$ be semisimple with eigenvalues

$$\underbrace{a'_1, \ldots, a'_{n'}, 1, (a'_{n'})^{-1}, \ldots, (a'_1)^{-1}}_{\text{from } \gamma'}, \underbrace{a''_1, \ldots, a''_{n''}, 1, (a''_{n''})^{-1}, \ldots, (a''_1)^{-1}}_{\text{from } \gamma''}.$$

We say that $\delta \in G(F)$ correspond à γ if it is semisimple with eigenvalues

$$a'_1, \ldots, a'_{n'}, (a'_{n'})^{-1}, \ldots, (a'_1)^{-1}, -a''_1, \ldots, -a''_{n''}, -(a''_{n''})^{-1}, \ldots, -(a''_1)^{-1}.$$

Attention: the eigenvalues from γ'' are multiplied by -1.
It can be shown that every semisimple $\gamma \in H(F)$ correspond to some $\delta \in G(F)$.

Stability

Following Adams, we have a *ad hoc* definition of stable conjugacy on $\tilde{G}$. Define

$$\Phi := \frac{\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-}{|\cdots|}.$$

It is a smooth function on the preimage of $\{g \in G(F) : \det(g+1) \neq 0\} \supset G_{\mathrm{reg}}(F)$. It is genuine in the sense that $\Phi(zx) = z\Phi(x)$ for all $z \in \mu_8$.

Definition

Let $\tilde{\delta}_1, \tilde{\delta}_2 \in \tilde{G}$ be semisimple regular and δ_1, δ_2 their images in $G(F)$. They are called stably conjugate if

- δ_1, δ_2 are stably conjugate, and
- $\Phi(\tilde{\delta}_1) = \Phi(\tilde{\delta}_2)$.

This does have some explanations in the case $F = \mathbb{R}$. Moreover, the function Φ satisfies a product formula.

The transfer factor Δ

We want a function $\Delta : H_{G-\mathrm{reg}}(F) \times \tilde{G} \to \mathbb{C}$ such that $\Delta(\gamma, \tilde{\delta}) \neq 0$ iff γ and $\delta = p(\tilde{\delta})$ correspond. Adams defined the factor Δ in the case $n'' = 0$ by

$$\Delta(\gamma, \tilde{\delta}) = \Phi(\tilde{\delta}).$$

In general, suppose that $\gamma = (\gamma', \gamma'')$ is G-regular and corresponds to δ.
Then there is an orthogonal decomposition $W = W' \oplus W''$ according to the eigenvalues from γ' and γ''.
There is a canonical homomorphism $j : \widetilde{\mathrm{Sp}}(W') \times \widetilde{\mathrm{Sp}}(W'') \to \widetilde{\mathrm{Sp}}(W)$.
$\mathrm{Ker}(j)$ is μ_8 anti-diagonally embedded. One can choose $(\tilde{\delta}', \tilde{\delta}'')$ such that $j(\tilde{\delta}', \tilde{\delta}'') = \tilde{\delta}$.
Define $\Delta' := \frac{\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-}{|\cdots|}$, $\Delta'' := \frac{\mathrm{tr}\omega_\psi^+ + \mathrm{tr}\omega_\psi^-}{|\cdots|}$. Both are genuine.

Definition

Set

$$\Delta(\gamma, \tilde{\delta}) := \Delta'(\tilde{\delta}')\Delta''(\tilde{\delta}'')\Delta_0(\gamma, \delta)$$

where Δ_0 is some correction factor which is stably invariant.

Although Δ_0 is a relatively simple term, we will not write it down since this requires a parametrization of semisimple orbits.
Remark: When $F = \mathrm{R}$, the factor Δ_0 is that defined by Renard, albeit in a somehow different formalism.

Properties of Δ

1. Δ is genuine: $\Delta(\gamma, z\tilde{\delta}) = z\Delta(\gamma, \tilde{\delta})$ for all $z \in \mu_8$.
2. Cocycle condition.
3. Parabolic descent.
4. Normalization in the unramified case: $\Delta(\gamma, \delta) = 1$ if γ, δ correspond, are in hyperspecial subgroups and have regular reductions.
5. Product formula: F global, then $\prod_v \Delta_v(\gamma, \delta) = 1$ if $\gamma \in H(F)$ correspond $\delta \in G(F)$.
6. Symmetry: $\Delta_{n',n''}((\gamma', \gamma''), \tilde{\delta}) = \Delta_{n'',n'}((\gamma'', \gamma'), (-1) \cdot \tilde{\delta})$, where $-1 \in \tilde{G}$ is some appropriate preimage of $-1 \in G(F)$.
7. Semisimple descent.

Conjectures à la Langlands-Shelstad

Suppose that $\gamma \in H(F)$ is G-regular and $f \in C_c^\infty(\tilde{G})$ is anti-genuine. Define the endoscopic orbital integral of f by

$$J_{\tilde{G},H}(\gamma, f) := \sum_{\delta} \Delta(\gamma, \tilde{\delta}) J_{\tilde{G}}(\tilde{\delta}, f)$$

where δ ranges over the conjugacy classes in $G(F)$ corresponding to γ. This is independent of the choice of $\tilde{\delta} \in \tilde{G}$ over δ. Also note that $J_{\tilde{G}}$ is an integral over $G_\delta(F)\backslash G(F)$ since two elements in $\tilde{G}$ commute iff their images in $G(F)$ commute.

Definition

We say $f^H \in C_c^\infty(H(F))$ is a transfer of f if $J_{\tilde{G},H}(\gamma, f) = J_H^{st}(\gamma, f^H)$ for all G-regular γ, where we use matching Haar measures on $H_\gamma(F) \simeq G_\delta(F)$.

Transfer theorem

For any f, the transfer f^H always exists.

Suppose that F is local, nonarchimedean of residual characteristic p large enough. Also suppose that $\psi|_{\mathfrak{o}_F} = 1$, $\psi|_{\mathfrak{p}_F^{-1}} \neq 1$. Let K be a hyperspecial subgroup of $G(F)$, regarded as a subgroup of $\tilde{G}$. Let f_K be the anti-genuine function such that $f_K(zK) = z^{-1}$ and $f_K = 0$ outside $\mu_8 K$.

Fundamental lemma for units

One can take f_K^H to be $\mathbb{1}_{K_H}$ where K_H is any hyperspecial subgroup of $H(F)$. Here we use measures such that K and K_H have mass 1.

Following Kottwitz's approach, one can stabilize the regular elliptic terms of the geometric side of the trace formula, applied to anti-genuine test functions.

Remark. (1) The real case is solved by Renard by using a characterization à la Shelstad of stable orbital integrals on $\tilde{G}$. The complex case is almost trivial. (2) For archimedean F, one can also consider the transfer of Schwarz functions.

Descent to Lie algebra

F: local field, M: reductive group over F. Set $M^\eta := Z_M(\eta)$ for any $\eta \in M(F)$.
For every semisimple $\eta \in M(F)$ and a $M^\eta(F)$-invariant open set $\mathfrak{U}' \subset \mathfrak{m}_\eta(F)$ containing 0, there exists a $M^\eta(F)$-invariant open set $\mathfrak{U} \subset \mathfrak{U}'$ such that

1. $\exp : \mathfrak{U} \to M_\eta(F)$ is a homeomorphism onto its image, which is open;
2. $(X, x) \mapsto x^{-1} \exp(X) \eta x$ from $\mathfrak{U} \times M(F)$ to $M(F)$ is submersive, its image $\mathfrak{U}^\natural$ is open and $M^\eta(F)$-invariant;
3. if $x \in M(F)$ and if $X \in \mathfrak{U}$ is such that $x^{-1} \exp(X) \eta x \in \exp(\mathfrak{U})\eta$, then $x \in M^\eta(F)$.

The same holds if M is replaced by some nonlinear covering of reductive groups.
If M is reductive and $\mathfrak{U}'$ is stably invariant, we may assume that $\mathfrak{U}$ is also stably invariant, and (3) holds for $x \in M(\bar{F})$ (resp. $x \in M^\eta(\bar{F})$).

Let $\mathfrak{U}$ be as above. M is a reductive group over F or a covering. Suppose that $\mathfrak{U}$ is small enough.

- For any $f^\natural \in C_c^\infty(\mathfrak{U}^\natural)$, there exists $f \in C_c^\infty(\mathfrak{U})$ such that

$$J_{M_\eta}(X, f) = J_M(\exp(X)\eta, f^\natural)$$

  for all $X \in \mathfrak{U}_{\mathrm{reg}}$.
- Conversely, for any $f \in C_c^\infty(\mathfrak{U})$ such that $X \mapsto J_{M_\eta}(X, f)$ is invariant by $M^\eta(F)$, there exists $f^\natural \in C_c^\infty(\mathfrak{U}^\natural)$ satisfying the equality above.

Suppose that M is reductive and $\mathfrak{U}$ is invariant by conjugaison by $M^\eta(\bar{F})$, then the statements above are still valid for the equality

$$J^{\mathrm{st}}_{M_\eta}(X, f) = J^{\mathrm{st}}_M(\exp(X)\eta, f^\natural),$$

and the condition on f for the existence of $f^\natural$ becomes: $X \mapsto J_{M_\eta}(X, f)$ is invariant by conjugacy by $M^\eta(\bar{F})$.

Endoscopy on Lie algebras

Suppose that F is nonarchimedean. By a characterization theorem of Langlands and Shelstad, the conjecture of transfer (resp. fundamental lemma - modulo some arithmetic results...) can be reduced similar questions for the Lie algebras $\mathfrak{g}_\eta$, $\mathfrak{h}_\epsilon$, where $\eta \in G(F)$ and $\epsilon \in H(F)$ are corresponding semisimple elements such that H_ϵ is quasisplit.
For the sake of simplicity, assume $n'' = 0$ so that $H = \mathrm{SO}(2n+1)$ and consider the following two cases:

1. $\eta = 1, \epsilon = 1$;
2. $\eta = -1, \epsilon = \begin{pmatrix} -1 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 \\ 0 & \dots & -1 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$.

In either case, define the local transfer factor $\Delta^\flat : \mathfrak{h}_\epsilon(F) \times \mathfrak{g}_\eta(F) \to \mathbb{C}$ by setting

$$\Delta^\flat(Y, X) = \Delta(\exp(Y)\epsilon, \exp(X)\tilde{\eta})$$

where $\tilde{\eta}$ is any chosen preimage in $\tilde{G}$ of η. This is originally defined for small X, Y, but can be extended by a homogeneity result.

Case 1: a nonstandard situation

In this case, $\exp(Y)$ corresponds to $\exp(X)$ iff Y and X correspond by eigenvalues.

Fact (follows from Maktouf's character formula)

$\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-$ is a nonzero constant near 1. Hence so is $\Delta^\flat(Y, X)$ for corresponding elements Y, X.

Our task is to match stable orbital integrals on the Lie algebras of $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$, w.r.t. the point correspondence by eigenvalues.
This is NOT accounted by the theory of endoscopy! Orz
What saves the day is the **nonstandard endoscopy**, studied systematically by Waldspurger to show that *L'endoscopie tordue n'est pas si tordue*. We need to apply nonstandard transfer (resp. nonstandard fundamental lemma) to the triple $(\mathrm{Spin}(2n+1), \mathrm{Sp}(2n), \ldots)$.

Case 2: Local behavior of the character of the Weil representation

As before, $\exp(Y)\epsilon$ corresponds to $\exp(X)\eta$ iff Y and X correspond by eigenvalues. We have $G_\eta = G = \mathrm{Sp}(W)$ and $H_\epsilon = \mathrm{SO}(V, q)$ (quasisplit) where $\dim_F V = 2n$ and q is a quadratic form over F.
We want to match:

1. stable orbital integrals on the Lie algebra of $\mathrm{SO}(V, q)$;
2. endoscopic orbital integrals on the Lie algebras for the pair $(\mathrm{Sp}(W), \mathrm{SO}(V, q))$, defined w.r.t. the transfer factor $\Delta^\flat$.

Observation: $\mathrm{SO}(V, q)$ is an elliptic endoscopic group of $\mathrm{Sp}(W)$, the point correspondence is exactly the correspondence by eigenvalues. On the other hand, Maktouf's formula shows that the phase of $\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-$ oscillates in some mysterious manner near -1. The upshot is to show the

Fact

$\Delta^\flat$ is, up to a harmless constant, the transfer factor for the endoscopic group $\mathrm{SO}(V, q)$ of $\mathrm{Sp}(W)$.

It turns out that the oscillation of $\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-$ near singular points creates the main technical difficulty. On the contrary, the phase of $\mathrm{tr}\omega_\psi^+ + \mathrm{tr}\omega_\psi^-$ is constant near -1 but oscillates near 1. Indeed, by choosing an appropriate preimage of -1 in $\tilde{G}$, still denoted by -1, we have

$$(\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-)(\tilde{x}) = (\mathrm{tr}\omega_\psi^+ + \mathrm{tr}\omega_\psi^-)((-1)\cdot\tilde{x})$$

for all $\tilde{x} \in \tilde{G}$ with image $\delta \in G(F)$ satisfying $\det(\delta+1) \neq 0$.
Using Maktouf's character formula and some combinatorics on the lagrangian Grassmannian associated to $(W, \langle|\rangle)$, one can prove the following

Theorem

For $\tilde{x} \in \tilde{G}$ with image $x \in G(F)$ such that $\det(x \pm 1) \neq 0$, we have

$$\left(\frac{\mathrm{tr}\omega_\psi^+ + \mathrm{tr}\omega_\psi^-}{\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-}\right)(\tilde{x}) = \gamma_\psi(q[C_x]) \cdot \left|\frac{\det(x+1)}{\det(x-1)}\right|^{\frac{1}{2}}$$

where $C_x := 2 \cdot \frac{x-1}{x+1} \in \mathfrak{sp}(W)$, and $q[C_x]$ is the quadratic form on W defined by $(w_1, w_2) \mapsto \langle C_x w_1 | w_2 \rangle$.

One can show that $q[C_{\exp(X)}]$ is isometric to $q[X]$ when X is small enough or when X is topologically nilpotent and p is large enough. Hence the phase of $(\mathrm{tr}\omega_\psi^+ - \mathrm{tr}\omega_\psi^-)(-\exp(X))$ is, up to some harmless constant, equal to $\gamma_\psi(q[X])$. Now we need to study two things.

1. The transfer factor on Lie algebras for $(\mathrm{SO}(V, q), \mathrm{Sp}(W))$: this is explicitly calculated by Waldspurger in terms of an explicit parametrization of semisimple conjugacy classes in classical groups.
2. The Weil constant of the quadratic form $q[X]$. By using the same parametrization, this can be reduced to a tricky exercice on quadratic forms.

Let T be the centralizer of X, it is a maximal torus in G. The conjugacy classes in the stable conjugacy class containing X is a torser under $H^1(F, T)$ since G is semisimple and simply connected.
We show that the two terms vary in the same manner under the action of $H^1(F, T)$. By choosing a good base point, we show that (1), (2) differs at most by a term $\gamma_\psi(\det X)$.
However, $\det X = \det Y$ since they correspond by eigenvalues. Using Pfaffians, it's another exercice that $\det Y$ only depends on the quadratic form (V, q). Thus (1) and (2) differ by a constant, as asserted.

Outlook

- Fundamental lemma for spherical anti-genuine functions - it suffices to use a variant of Hales' proof and reduce to the case of units.
- Stabilization of the elliptic (possibly singular) terms in the geometric side of the trace formula - use semisimple descent and germ expansions following Langlands and Shelstad.
- Stabilization of the full trace formula: need the weighted fundamental lemma and many other things. This seems to be accessible in view of the recent work of Chaudouard and Laumon.
- Applications to representation theory: classification of anti-genuine (or equivalently, genuine) admissible irreducible representations into packets.