Langlands Programme: review and outlook

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Abstract

These are notes for a talk at the Taida Institute of Mathematical Sciences (TIMS) on February 13, 2015 at Taipei. I try to give a quick, personal and biased survey of some aspects of Langlands program. Details are intentionally omitted.

Nowadays the Langlands program is widely known among mathematicians. However, this huge theory is undergoing a substantial evolution that is largely unknown to outsiders. In this short note, we shall give a quick sketch thereof, although often inaccurate (with apologies), and give pointers to the literature. For various reasons, we will not touch on the motivic aspect, the function field case or the geometric Langlands despite their importance.

The reader is assumed to have some acquaintance of representations and automorphic forms. For more systematic overviews on Langlands program, we recommend [26, 25, 17], just to mention a few.

1 Recollections

Generalities Consider a topological space X equipped with a Radon measure. The central issue in harmonic analysis is to decompose the space $L^2(X)$ under a certain group action. Examples:

- $X = \mathbb{R}^n$, which boils down to the L^2-theory of Fourier analysis on euclidean spaces;
- $X = (\mathbb{R}/\mathbb{Z})^n$: Fourier analysis on tori — equally familiar;
- $X = \Gamma\backslash G$ where G is a real Lie group and Γ is a discrete subgroup thereof: this leads to the classical formulation of automorphic forms;
- (*The group case*) $X = G(F)$ where F is a local field (eg. $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$) and G is a connected reductive F-group: this is the L^2-setup for Plancherel formula for the locally compact group $G(F)$.

In each case, there is a group acting unitarily on the left of $L^2(X)$: for example, $(x, y) \in (G \times G)(F)$ acts on $L^2(X)$ by $\phi(\bullet) \mapsto \phi(x^{-1} \bullet y)$. Under mild conditions on X, there exists an essentially unique decomposition into a *direct integral*

$$L^2(X) \xrightarrow{\sim} \int^{\oplus}_{\Pi_{\mathrm{unit}}(G)} \mathcal{H}_\pi \, \mathrm{d}\mu(\pi)$$

of representations on Hilbert spaces, where

- $\Pi_{\mathrm{unit}}(G)$ is the *unitary dual* of $G(F)$, equipped with the Fell topology;
- μ is the *Plancherel measure* associated to $(G(F), L^2(X))$.

See [13] or [37, Chapter 14] for details.

After the middle of the 20th century, the focus has shifted to finding an *explicit description* of the Plancherel decomposition above. For example: what is the support of μ? what are the "atoms' of $(\Pi_{\mathrm{unit}}(G), \mu)$? Let's focus on the group case. Just as we have $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}'$ (an instance of rigged Hilbert space or Gelfand triple [16, Chapter 1], [6]) in classical Fourier analysis, it turns out that there is a convenient space of "test functions" $\mathcal{C}(G)$ such that $\mathcal{C}(G) \subset L^2(G(F)) \subset \mathcal{C}(G)'$, on which the $\int^{\oplus}$-decomposition can precisely stated. Harish-Chandra developed the whole machinery of $\mathcal{C}(G)$, constant terms, intertwining operators, c-functions etc., to give an explicit decomposition of $L^2(G(F))$ as a $(G \times G)(F)$-representation. The "atoms" are the *discrete series* whereas $\mathrm{Supp}(\mu) = \Pi_{\mathrm{temp}}(G)$, the *tempered dual* of $G(F)$.

Automorphic setting Let F be a global field with adèle ring $\mathbb{A}$, and let G be a connected reductive F-group. The modern setting for automorphic forms is the space

$$X := G(F)\backslash G(\mathbb{A})$$

on which $G(\mathbb{A})$ acts by right translation. Define $\mathcal{A}(X)$ to be the space of functions $\phi : X \to \mathbb{C}$ satisfying

(i) ϕ is C^∞;
(ii) ϕ and its derivatives under $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ have *moderate growth*, i.e. dominated by some polynomial of a height function $\|\cdot\|$ on $G(\mathbb{A})$ defined in terms of matrix entries (also called the *algebraic scale structure* in [6]);
(iii) ϕ is $\mathfrak{z}$-finite when $\mathrm{char}(F) = 0$, where $\mathfrak{z} := Z(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$;
(iv) ϕ is K-finite under bilateral translation, where K is some suitable maximal compact subgroup of $G(\mathbb{A})$.

Note that the K-finiteness can sometimes be removed. One often considers the quotient

$$X := G(F)A_{G,\infty}\backslash G(\mathbb{A})$$

for some central subgroup $A_{G,\infty}$ so that $\mathrm{vol}(X) < +\infty$ (reduction theory), and works within the space $\mathcal{A}^2(X) := L^2(X) \cap \mathcal{A}(X)$ of L^2-automorphic forms. Subquotients of $L^2(X)$ are called the (L^2-) automorphic representations. They are generated by L^2-automorphic forms.

To remedy the perplexities on (ii), we simply quote from Harish-Chandra:

"Without the condition of moderate growth you can't do anything."

Also note that the adélic approach does not capture all the richness of classical automorphic forms. By the way, it is often necessary to consider coverings $\tilde{G} \twoheadrightarrow G(\mathbb{A})$ in many circumstances (eg. study of θ-functions). A reasonable family of covering groups to work with is those arising from *Brylinski-Deligne extensions* [9]. For these coverings, multiplication in $\tilde{G}$ involves not only the algebraic structure of G, but also certain K-theoretical aspects of F.

2 Langlands' insights

Put $X_G := G(F)A_{G,\infty}\backslash G(\mathbb{A})$ as above. In what follows, the representations are implicitly assumed to be smooth and admissible. At the archimedean places, it is customary to work with Harish-Chandra modules (i.e. $(\mathfrak{g}, K)$-modules) or smooth Fréchet representations of moderate growth (Casselman-Wallach): see [7] for a systematic discussion.

Under these conventions, an irreducible representation π of $G(\mathbb{A})$ factorizes into a *restricted tensor product* $\bigotimes'_v \pi_v$ by Flath's theorem. Therefore, the study of automorphic representations divides into two stages.

- Study of $G(F_v)$-representations at each place v of F, which certainly has an independent interest;
- Determine the multiplicity of an irreducible representation $\pi = \bigotimes'_v \pi_v$ of $G(\mathbb{A})$ in $L^2(X_G)$ — this part has deep connections with arithmetic!

One defines the Langlands dual group $G^\vee$ of G as a connected reductive group over $\mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$ for some prime ℓ; to make this canonical one should fix a *pinning* of G, and then $G^\vee$ admits a Galois action via *pinned automorphisms* (see [12, A.4]). Define the L-group of G as ${}^L G = G^\vee \rtimes W_F$, where W_F stands for the Weil-group of F. Same for groups over the local fields F_v.

Local correspondence In the local case $G_v = G(F_v)$, the question is reduced to describing $\Pi_{\text{temp}}(G_v)$, via Langlands quotients [27]. The *local Langlands correspondence* predicts a surjection

$$\Pi_{\text{temp}}(G_v) \twoheadrightarrow \Phi_{\text{bdd}}(G_v) := \left\{\text{bounded } L\text{-parameters } \phi : W'_{F_v} \to {}^L G\right\} \Big/ G^\vee - \text{conj}$$

with various properties (eg. compatibility with normalized parabolic induction); for the case of classical groups, see [4, Theorem 1.5.1]. Recall that

$$W'_{F_v} := \begin{cases} W_{F_v} \times \mathrm{SU}(2), & F_v \text{ is non-archimedean}, \\ W_{F_v}, & F_v \text{ is archimedean}. \end{cases}$$

is the *Weil-Deligne group*. Its fibers Π_ϕ are called the (tempered) L-packets, which are expected to be finite and whose "internal structure" should be controlled by the $\mathcal{S}$-group

$$\mathcal{S}_\phi := \pi_0\left(Z_{G^\vee}(\operatorname{Im}\phi), 1\right) \Big/ Z_{G^\vee}^{\text{Gal-inv}} \hookrightarrow (G^\vee)_{\text{ad}}$$

for G quasi-split over F_v; in general, one should pass to its inverse image in $(G^\vee)_{\text{sc}}$ (this is rather subtle: see [1]). This is connected with the phenomenon of *endoscopy*.

To pass to non-tempered case, one can either (i) use Langlands quotients to reduce to tempered ones, or (ii) replace W'_{F_v} by $W'_{F_v} \times \mathrm{SU}(2)$ and work with Arthur's A-packets.

The A-packets are related to global questions and endoscopic character identities; many questions about A-packets remain unsolved even over $\mathbb{R}$ and $\mathbb{C}$, for example their unitarizability. For the conjectures, see [5].

When F is archimedean, the tempered local Langlands correspondence is largely based on Harish-Chandra's work on the tempered spectrum. For non-archimedean F, local correspondences have been established in many cases, such as GL_n (Henniart, Harris-Taylor, or Scholze), the classical groups (Arthur [4], Mok, Kaletha-Minguez-Shin-White via trace formula), SL_n and its inner forms (Hiraga-Saito [20]), etc., under various conditions.

Global correspondence Over a global field F, one expects a similar correspondence with the conjectural *automorphic Langlands group* $\mathcal{L}_F$ instead of W_F; it is equipped with a continuous homomorphism $\mathcal{L}_F \to W_F$. The existence of $\mathcal{L}_F$ hinges upon certain Tannakian structures on the automorphic representations of $\mathrm{GL}(n)$, for various n. Currently the existence of $\mathcal{L}_F$ seems out of reach, and it might turn out to be the last theorem to be proven in Langlands program! there do exist some speculations and substitutes for $\mathcal{L}_F$, however: see [2] and [4, §8.5].

Also note that M. Weissman [38] is currently developing a Langlands program for the covering groups coming from Brylinski-Deligne extensions.

Functoriality In both local and global cases, for connected reductive groups H, G and a homomorphism ${}^L H \to {}^L G$ such that the diagram

$$\begin{array}{ccc} {}^L H & \longrightarrow & {}^L G \\ & \searrow \quad \swarrow & \\ & W_F & \end{array}$$

commutes, the Langlands correspondence suggests that one may "lift" a packet of representations from H to G. This *principle of functoriality* can be formulated without presuming Langlands correspondence, and they are actually interlocked in many circumstances, for instance: the use of base change in proving automorphy of certain Galois representations.

Functoriality is extremely powerful. For example, the existence of arbitrary symmetric-power lifts will imply the generalized Ramanujan conjecture, which is commonly believed to be as hard as the functoriality in general! We refer to the long survey [8].

3 Some approaches

Converse theorems Weil's converse theorem characterizes the L-functions $L(s,f) = \sum_{n \geq 0} a_n n^{-s}$ coming from a holomorphic modular form f in terms of the analytic properties and functional equations of its twists

$$L(s, f \times \chi) = \sum_{n \geq 0} a_n \chi(n) n^{-s}$$

by Dirichlet characters. Generalization to general GL_n is obtained in a long series of papers by Piatetski-Shapiro and his collaborators.

Let $\pi = \bigotimes'_v \pi_v$ be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$. Assume that the central character ω_π is an idèle class character. Form the partial L-function $L^S(s,\pi)$ as an Euler product and assume its convergence for $\mathrm{Re}(s) \gg 0$. Roughly speaking, the converse theorem in [10] asserts that π is cuspidal automorphic if for all $1 \leq m \leq n-1$ and all cuspidal automorphic representation τ of $\mathrm{GL}_m(\mathbb{A})$, the *Rankin-Selberg L-function* [22] $L(s,\pi\times\tau)$, as an Euler product, satisfies *nice analytic properties* as follows:

(i) $L(s,\pi\times\tau)$ extends to an entire function on $\mathbb{C}$;

(ii) $L(s,\pi\times\tau)$ is bounded in vertical strips of finite width;

(iii) $L(s,\pi\times\tau)$ satisfies a functional equation $s \leftrightarrow 1-s$ with respect to suitable ε-factors.

The converse theorem is useful for establishing functorial lifts to GL_n; it also plays a crucial rôle in the proof [29] of the global Langlands correspondence for GL_n over function fields. The main obstacle is the niceness alluded to above, which often requires ingenuous *zeta integrals* representing the relevant L-functions — one may regard the Langlands-Shahidi method [35] as an instance, in which the analytic properties come ultimately from those of Eisenstein series......

Another issue is to limit the ramification and the rank m of the twisting representation τ; for example, Jacquet conjectures that $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$ suffices. We refer to Cogdell's survey [11] on these questions.

Trace formula with endoscopy The comparison of Arthur-Selberg trace formulae is capable of yielding quite complete results about automorphic representations. Thus far, in characteristic zero it is most successful in the case of classical groups [4] via *twisted endoscopy*. Even for classical groups, the prerequisites are overwhelming at first sight: twisted trace formula, endoscopic transfer, fundamental lemma, stabilization, etc. The remaining ingredients for [4] are now completed in a really long series of preprints by Mœglin and Waldspurger.

Take the split group SO_{2n+1} over a local field F for example, its dual group is $\mathrm{Sp}_{2n}(\mathbb{C})$ with trivial Galois action, which may be realized as

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{x \in \mathrm{GL}_{2n}(\mathbb{C}) : \theta(x) = x\}$$

where we choose a symplectic form $\langle\cdot|\cdot\rangle$ on $\mathbb{C}^{2n}$ and define the involution θ of GL_{2n} by

$$\langle x(w_1)|w_2\rangle = \langle w_1|\theta(x)^{-1}(w_2)\rangle, \quad w_1, w_2 \in \mathbb{C}^n.$$

Simply put, θ is the "transpose-inverse" with respect to $\langle\cdot|\cdot\rangle$. Very roughly speaking, twisted endoscopy [28] allows one to compare

$$\{\text{irreps of } \mathrm{SO}_{2n+1}(F)\} \quad \text{and} \quad \{\text{twisted irreps of } \mathrm{GL}_{2n}(F)\}$$

in a manner dual to the inclusion $\mathrm{Sp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$ of fixed-points of the involution θ. Here, irreducible "twisted representation" may be viewed as a pair $(\tilde{\pi},\pi)$ where (π,V) is an irreducible representation of $\mathrm{GL}_{2n}(F)$ and $\tilde{\pi}$ is a map from the $\mathrm{GL}_{2n}(F)$-bitorsor of non-degenerate bilinear forms on F^{2n} to $\mathrm{End}_{\mathbb{C}}(V)$ such that

$$\tilde{\pi}(gxh) = \pi(g)\tilde{\pi}(x)\pi(h), \quad g,h \in \mathrm{GL}_{2n}(F).$$

Now the other cases should become reasonable: it remains to incorporate the Galois action carefully for non-split SO and unitary groups. People are always thinking about a similar strategy for the exceptional group G_2; I am not sure about the current status of this project.

The L-function machine Since we have mentioned the zeta integrals. It is time to sketch the well-known paradigm of integral representations of L-functions. The simplest case is the global zeta integral in Tate's thesis

$$Z(\chi|\cdot|^s, \Phi) = \int_{\mathbb{A}^\times} \chi(x)|x|^s\Phi(x)\,\mathrm{d}^\times x, \quad \Phi = \prod_v \Phi_v \in \mathcal{S}(\mathbb{A})$$

for $\mathrm{Re}(s) \gg 0$ and $\chi = \prod_v \chi_v : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$; it "represents" the Hecke L-function $L(s,\chi)$. In general, the machine functions in five steps.

1. Write done the appropriate global zeta integral $Z(s,\pi,\cdots)$ — truly an art, and express it as an Euler product $\prod_v Z(s,\pi_v,\cdots)$ at least when $\mathrm{Re}(s) \gg 0$.

2. Establish the analytic properties of global zeta integrals (meromorphic continuation, etc.)

3. Establish the analytic properties of local zeta integrals $Z(s,\pi_v,\cdots)$.

4. Unramified calculation: show that for a certain "basic" test function at a unramified place, $Z(s,\pi_v,\cdots)$ equals the desired unramified $L(s,\pi_v,\rho)$.

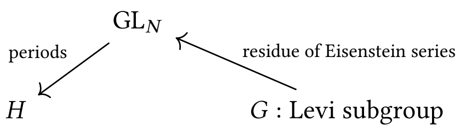
5. Establish other properties of local zeta integrals such as non-vanishing, and define local L-factors $L(s,\pi_v,\rho)$ at each place v as greatest common divisors.

Once successful, the machine will output analytic properties of the partial automorphic L-function $L^S(s,\pi,\rho)$.

Automorphic descent Unlike the approaches via converse theorem or trace formula, the *descent method*, previously known as backward lifting (!!), gives a more explicit construction of automorphic forms. We refer to the book [19] for a systematic introduction. Roughly speaking, one proceeds by first taking a suitable residue of an Eisenstein series on a big group $\mathrm{GL}_N(\mathbb{A})$, induced from a cuspidal automorphic representation τ on a Levi subgroup $G(\mathbb{A})$, and then taking suitable periods to obtain automorphic forms on smaller classical groups $H(\mathbb{A})$. The relevant periods here are of

- Bessel/Gelfand-Graev type (orthogonal case), or
- Fourier-Jacobi type (symplectic or metaplectic case).

Integral representations of this sort date back to the works of Ginzburg, Piatetski-Shapiro *et al* [18]. Etymology: descent is an operation inverse to functoriality lift.



There is also a local counterpart of descent. They are intricately connected with questions in relative harmonic analysis such as multiplicity one of Bessel or Fourier-Jacobi models, the Gan-Gross-Prasad conjectures, etc. Currently, this approach is being developed rapidly by Professor D. Jiang and his school.

4 The relative setup

To begin with, assume F local. In our earlier digression about decomposing $L^2(X)$, it should be clear that *a priori*, there is no reason to limit to the group case $X = G(F)$. One can also consider "reasonable" spaces X with a right $G(F)$-action and study the spectrum of $L^2(X)$. Note that we pretend X equipped with an invariant measure: in general, there is always some workaround...

Several natural questions arise:

(i) When does an irreducible unitary representation of $G(F)$ appears in $L^2(X)$? Alternatively, when does an admissible irreducible $G(F)$-representation π embed into $C^\infty(X)$? Note that the second question in the non-archimedean group case is easy: an admissible irreducible representation Π of $G(F) \times G(F)$ embeds into $C^\infty(G(F))$ if and only if $\Pi \simeq \check{\pi} \boxtimes \pi$ (here $\check{\pi}$ is the contragredient), in which case the space of embeddings is 1-dimensional; indeed, it is spanned by the matrix coefficient map

$$\check{v} \otimes v \longmapsto \langle \check{v}, \pi(\cdot) v \rangle \in C^\infty(X).$$

(ii) Describe the space of embeddings $\mathrm{Hom}_G(\pi, C^\infty(X))$ mentioned above. Under what conditions is it finite-dimensional?

(iii) Describe the Plancherel measure for the direct integral decomposition of $L^2(X)$.

(iv) Explicate its relation with local Langlands correspondence, A-packets, etc.

Note that when F is non-archimedean and $X = H(F)\backslash G(F)$, the space $\mathrm{Hom}_G(\pi, C^\infty(X))$ is canonically isomorphic with $\mathrm{Hom}_H(\pi, \mathbb{C})$, the space of $H(F)$-invariant linear functionals. Representations with $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0$ are called *H-distinguished*. One must keep in mind that $(H\backslash G)(F) \supsetneq H(F)\backslash G(F)$ in general.

Now assume F global. One asks if the *H-period integral*

$$\phi \mapsto \int_{H(F)\backslash H(\mathbb{A})} \phi$$

(assuming convergence) of an automorphic representation π is identically zero. Automorphic representations π with nonzero H-periods are called globally H-distinguished. Distinction by a subgroup H is a central concern for automorphic representation. Global distinction implies local distinction everywhere, but the converse often requires serious global obstructions such as the non-vanishing of certain L-values. A well-known instance is the study of toric periods on PGL_2 by Waldspurger *et al* (see below).

The *relative trace formula* developed by Jacquet. King Fai Lai, Rallis *et al.* [21, 23, 24] is a powerful tool for studying periods. In particular, Waldspurger's work on toric periods has been treated in this way in [23].

These questions were studied on a case-by-case basis, eg. the algebraic symmetric spaces $H\backslash G$ where $(G^\theta)^0 \subset H \subset G^\theta$ for some involution $\theta : G \to G$. Recall that a *spherical variety* under a group G (let's assume it split) is a G-variety X which has an open orbit under a Borel subgroup $B \subset G$. The Luna-Vust theory provides a combinatorial description of spherical varieties via *colored cones*, which generalizes the familiar case of toric varieties; see eg. [36]. After the groundbreaking works of Sakellaridis (eg. [33, 34], also inspired by [15]), it seems that spherical varieties form a natural framework for relative harmonic analysis. Furthermore, it is possible to attach L-groups to a spherical homogeneous space, thereby integrating the relative theory into functoriality.

This is a highly active field of research and it is beyond my capability to give a survey. Nevertheless, they are likely to become a must-read for future generations.

5 Beyond endoscopy?

So far, most instances of the functoriality are obtained for L-embeddings ${}^L H \hookrightarrow {}^L G$ of L-groups which are not "too different" (eg. the endoscopic case). In [31, 14] Langlands proposed a new usage of the trace formula to prove new cases of functoriality. Being a novice, I can only outline some facets of the program(s). His ideas seem to involve the following:

- A notion of primitive automorphic representations of G (= not lifted from a smaller H), to be detected by poles of L-functions.
- Study poles of L-functions by inserting carefully crafted test functions into the trace formula, then consider some limit form of the cuspidal part with analytic tools.
- Use of the Poisson summation formula or its variants (cf. Lafforgue's "kernel for functoriality" [30]).
- Work with the *stable trace formula* [3, §29] that seems indispensable for comparing trace formulae.
- Non-endoscopic stable character relations, and a study of transfer factors, singularities, etc. in this framework [32]

See also the afterward of [3] for relevant discussions.

Surely, this is not the end of the story. We recommend Chung Pang Mok's slides on Speculations about Langlands program.

References

[1] James Arthur. "A note on L-packets". In: *Pure Appl. Math. Q.* 2.1, Special Issue: In honor of John H. Coates. Part 1 (2006), pp. 199–217. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2006.v2.n1.a9. URL: http://dx.doi.org/10.4310/PAMQ.2006.v2.n1.a9.

[2] James Arthur. "A note on the automorphic Langlands group". In: *Canad. Math. Bull.* 45.4 (2002). Dedicated to Robert V. Moody, pp. 466–482. ISSN: 0008-4395. DOI: 10.4153/CMB-2002-049-1. URL: http://dx.doi.org/10.4153/CMB-2002-049-1.

[3] James Arthur. "An introduction to the trace formula". In: *Harmonic analysis, the trace formula, and Shimura varieties.* Vol. 4. Clay Math. Proc. Providence, RI: Amer. Math. Soc., 2005, pp. 1–263.

[4] James Arthur. *The endoscopic classification of representations*. Vol. 61. American Mathematical Society Colloquium Publications. Orthogonal and symplectic groups. American Mathematical Society, Providence, RI, 2013, pp. xviii+590. ISBN: 978-0-8218-4990-3.

[5] James Arthur. “Unipotent automorphic representations: conjectures”. In: *Astérisque* 171-172 (1989). Orbites unipotentes et représentations, II, pp. 13–71. ISSN: 0303-1179.

[6] Joseph N. Bernstein. “On the support of Plancherel measure”. In: *J. Geom. Phys.* 5.4 (1988), 663–710 (1989). ISSN: 0393-0440. DOI: 10.1016/0393-0440(88)90024-1. URL: http://dx.doi.org/10.1016/0393-0440(88)90024-1.

[7] Joseph Bernstein and Bernhard Krötz. “Smooth Fréchet globalizations of Harish-Chandra modules”. In: *Israel J. Math.* 199.1 (2014), pp. 45–111. ISSN: 0021-2172. DOI: 10.1007/s11856-013-0056-1. URL: http://dx.doi.org/10.1007/s11856-013-0056-1.

[8] Valentin Blomer and Farrell Brumley. “The role of the Ramanujan conjecture in analytic number theory”. In: *Bull. Amer. Math. Soc. (N.S.)* 50.2 (2013), pp. 267–320. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-2013-01404-6. URL: http://dx.doi.org/10.1090/S0273-0979-2013-01404-6.

[9] Jean-Luc Brylinski and Pierre Deligne. “Central extensions of reductive groups by $\mathbf{K}_2$”. In: *Publ. Math. Inst. Hautes Études Sci.* 94 (2001), pp. 5–85. ISSN: 0073-8301. DOI: 10.1007/s10240-001-8192-2. URL: http://dx.doi.org/10.1007/s10240-001-8192-2.

[10] J. W. Cogdell and I. I. Piatetski-Shapiro. “Converse theorems for GL_n”. In: *Inst. Hautes Études Sci. Publ. Math.* 79 (1994), pp. 157–214. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1994__79__157_0.

[11] J.W. Cogdell. “Piatetski-Shapiro’s work on converse theorems.” English. In: *Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro. Proceedings of the conference, Yale University, New Haven, CT, USA, April 23–27, 2012.* Providence, RI: American Mathematical Society (AMS), 2014, pp. 31–49. ISBN: 978-0-8218-9394-4/pbk.

[12] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*. Vol. 17. New Mathematical Monographs. Cambridge University Press, Cambridge, 2010, pp. xx+533. ISBN: 978-0-521-19560-7. DOI: 10.1017/CBO9780511661143. URL: http://dx.doi.org/10.1017/CBO9780511661143.

[13] Jacques Dixmier. *von Neumann algebras*. Vol. 27. North-Holland Mathematical Library. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett. North-Holland Publishing Co., Amsterdam-New York, 1981, pp. xxxviii+437. ISBN: 0-444-86308-7.

[14] Edward Frenkel, Robert Langlands, and Báo Châu Ngô. “Formule des traces et fonctorialité: le début d’un programme”. In: *Ann. Sci. Math. Québec* 34.2 (2010), pp. 199–243. ISSN: 0707-9109.

[15] Dennis Gaitsgory and David Nadler. “Spherical varieties and Langlands duality”. In: *Mosc. Math. J.* 10.1 (2010), pp. 65–137, 271. ISSN: 1609-3321.

[16] I. M. Gel′fand and N. Ya. Vilenkin. *Generalized functions. Vol. 4.* Applications of harmonic analysis, Translated from the Russian by Amiel Feinstein. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1964 [1977], pp. xiv+384.

[17] Jayce Robert Getz. “An introduction to automorphic representations”. 2013. URL: http://www.math.duke.edu/~jgetz/aut_reps.pdf.

[18] D. Ginzburg, I. Piatetski-Shapiro, and S. Rallis. “L functions for the orthogonal group”. In: *Mem. Amer. Math. Soc.* 128.611 (1997), pp. viii+218. ISSN: 0065-9266. DOI: 10.1090/memo/0611. URL: http://dx.doi.org/10.1090/memo/0611.

[19] David Ginzburg, Stephen Rallis, and David Soudry. *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011, pp. x+339. ISBN: 978-981-4304-98-6; 981-4304-98-0. DOI: 10.1142/9789814304993. URL: http://dx.doi.org/10.1142/9789814304993.

[20] Kaoru Hiraga and Hiroshi Saito. “On L-packets for inner forms of SL_n”. In: *Mem. Amer. Math. Soc.* 215.1013 (2012), pp. vi+97. ISSN: 0065-9266. DOI: 10.1090/S0065-9266-2011-00642-8. URL: http://dx.doi.org/10.1090/S0065-9266-2011-00642-8.

[21] H. Jacquet and K. F. Lai. “A relative trace formula”. In: *Compositio Math.* 54.2 (1985), pp. 243–310. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1985__54_2_243_0.

[22] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika. "Rankin-Selberg convolutions". In: *Amer. J. Math.* 105.2 (1983), pp. 367–464. ISSN: 0002-9327. DOI: 10.2307/2374264. URL: http://dx.doi.org/10.2307/2374264.

[23] Hervé Jacquet. "Sur un résultat de Waldspurger. II". In: *Compositio Math.* 63.3 (1987), pp. 315–389. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1987__63_3_315_0.

[24] Hervé Jacquet, King F. Lai, and Stephen Rallis. "A trace formula for symmetric spaces". In: *Duke Math. J.* 70.2 (1993), pp. 305–372. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-93-07006-8. URL: http://dx.doi.org/10.1215/S0012-7094-93-07006-8.

[25] Lizhen Ji et al., eds. *Automorphic forms and the Langlands program.* Vol. 9. Advanced Lectures in Mathematics (ALM). Including papers from the International Conference "Langlands and Geometric Langlands Program" held in Guangzhou, June 18–21, 2007. International Press, Somerville, MA; Higher Education Press, Beijing, 2010, pp. viii+319. ISBN: 978-1-57146-141-4.

[26] A. W. Knapp. "Introduction to the Langlands program". In: *Representation theory and automorphic forms (Edinburgh, 1996).* Vol. 61. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1997, pp. 245–302. DOI: 10.1090/pspum/061/1476501. URL: http://dx.doi.org/10.1090/pspum/061/1476501.

[27] Takuya Konno. "A note on the Langlands classification and irreducibility of induced representations of p-adic groups". In: *Kyushu J. Math.* 57.2 (2003), pp. 383–409. ISSN: 1340-6116. DOI: 10.2206/kyushujm.57.383. URL: http://dx.doi.org/10.2206/kyushujm.57.383.

[28] Robert E. Kottwitz and Diana Shelstad. "Foundations of twisted endoscopy". In: *Astérisque* 255 (1999), pp. vi+190. ISSN: 0303-1179.

[29] Laurent Lafforgue. "Chtoucas de Drinfeld et correspondance de Langlands". In: *Invent. Math.* 147.1 (2002), pp. 1–241. ISSN: 0020-9910. DOI: 10.1007/s002220100174. URL: http://dx.doi.org/10.1007/s002220100174.

[30] Laurent Lafforgue. "Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires". In: *Jpn. J. Math.* 9.1 (2014), pp. 1–68. ISSN: 0289-2316. DOI: 10.1007/s11537-014-1274-y. URL: http://dx.doi.org/10.1007/s11537-014-1274-y.

[31] Robert P. Langlands. "Beyond endoscopy". In: *Contributions to automorphic forms, geometry, and number theory.* Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 611–697.

[32] Robert P. Langlands. "Singularités et transfert". In: *Ann. Math. Qué.* 37.2 (2013), pp. 173–253. ISSN: 2195-4755. DOI: 10.1007/s40316-013-0008-5. URL: http://dx.doi.org/10.1007/s40316-013-0008-5.

[33] Yiannis Sakellaridis. "On the unramified spectrum of spherical varieties over p-adic fields". In: *Compos. Math.* 144.4 (2008), pp. 978–1016. ISSN: 0010-437X. DOI: 10.1112/S0010437X08003485. URL: http://dx.doi.org/10.1112/S0010437X08003485.

[34] Yiannis Sakellaridis. "Spherical functions on spherical varieties". In: *Amer. J. Math.* 135.5 (2013), pp. 1291–1381. ISSN: 0002-9327. DOI: 10.1353/ajm.2013.0046. URL: http://dx.doi.org/10.1353/ajm.2013.0046.

[35] Freydoon Shahidi. *Eisenstein series and automorphic L-functions.* Vol. 58. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2010, pp. vi+210. ISBN: 978-0-8218-4989-7.

[36] Dmitry A. Timashev. *Homogeneous spaces and equivariant embeddings.* Vol. 138. Encyclopaedia of Mathematical Sciences. Invariant Theory and Algebraic Transformation Groups, 8. Heidelberg: Springer, 2011, pp. xxii+253. ISBN: 978-3-642-18398-0. DOI: 10.1007/978-3-642-18399-7. URL: http://dx.doi.org/10.1007/978-3-642-18399-7.

[37] Nolan R. Wallach. *Real reductive groups. II.* Vol. 132. Pure and Applied Mathematics. Boston, MA: Academic Press Inc., 1992, pp. xiv+454. ISBN: 0-12-732961-7.

[38] M. H. Weissman. "The L-group of a covering group". In: *ArXiv e-prints* (Jan. 2015). arXiv: 1501.06169 [math.NT].

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Last update: February 15, 2015