Full stable trace formula for $\widetilde{Sp}(2n)$

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May 15, 2021

May 24, 2021

Oct 20, 2021

The preprint is available on arXiv:2109.06581

This work is an outgrowth of my PhD thesis (2011). Sorry for the delay!

An incomplete list of works mentioned in this talk.

- J. Arthur, *A stable trace formula I—III*. (2002, 2001, 2003).
- J. Arthur, *The endoscopic classification of representations*, AMS Coll. Volume 61 (2013).
- W. T. Gan, A. Ichino, *The Shimura–Waldspurger correspondence for* Mp_{2n} (2018).
- L., *Transfert d'intégrales orbitales pour le groupe métaplectique* (2011)
- L., *La formule des traces stable pour le groupe métaplectique: les termes elliptiques* (2015)
- C. Luo, *Endoscopic character identities for metaplectic groups* (2020)
- C. Mœglin, J.-L. Waldspurger, *Stabilisation de la formule des traces tordue, Volume I, II*. Progress in Mathematics, 316—317 (2016).

What are automorphic representations?

They are far-reaching reinterpretations and generalizations of *modular forms*.

- F: number field, $\mathbb{A} = \mathbb{A}_F$: ring of adèles.
- G: connected reductive F-group.
- $L^2(G(F)\backslash G(\mathbb{A})^1) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}}$: the L^2-automorphic spectrum, $\mathrm{mes}(G(F)\backslash G(\mathbb{A})^1) < +\infty$.

Study of automorphic representations $\approx$ decomposition of $L^2(G(F)\backslash G(\mathbb{A})^1)$ under right regular $G(\mathbb{A})$-representation.

Arthur's Conjecture: $L^2(G(F)\backslash G(\mathbb{A})^1) = \hat{\bigoplus}_\psi L^2_\psi$,

- ψ ranges over Arthur parameters $\mathscr{L}_F \times \mathrm{SL}(2, \mathbb{C}) \to {}^{\mathrm{L}}G$,
- $\mathscr{L}_F$ is the hypothetical Langlands group of F.

What is the Arthur-Selberg trace formula?

Idea: access $L^2(G(F)\backslash G(\mathbb{A})^1)$ through an equality of invariant distributions on $G(\mathbb{A})$.

$$I^G_{\mathrm{geom}}(f) = I^G_{\mathrm{spec}}(f).$$

It is a far-reaching generalization of *Poisson summation formula*:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

where $f : \mathbb{R} \to \mathbb{C}$ is a function + growth conditions, and $\hat{f}$ is its Fourier transfer, suitably normalized.

Look at

$$I^G_{\mathrm{geom}}(f) = I^G_{\mathrm{spec}}(f).$$

Spectral side Main terms = sums of character-distributions $f \mapsto \operatorname{tr} \pi(f)$ where π are unitary irreducible representations of $G(\mathbb{A})$, weighted by their multiplicities $m(\pi)$ in L^2_{disc}.

Geometric side Main terms = sums of orbital integrals

$$f \mapsto \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\, \mathrm{d}g,$$

weighted by $\mathrm{mes}\left(G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1\right)$, where γ are elliptic regular semisimple orbits in $G(F)$ and $G_\gamma := Z_G(\gamma)^\circ$.

Example: Comparison of geometric sides for different groups $\rightsquigarrow$ cases of Langlands' Functoriality.

Structure of the trace formula

Non-compactness of $G(F)\backslash G(\mathbb{A})^1 \iff$ Existence of proper Levi subgroups $\iff$ Continuous spectrum in L^2.

Arthur's invariant trace formula: $I_{\text{geom}} = I_{\text{spec}}$

$$I^G = \sum_{\substack{M \supset M_0 \\ \text{Levi}}} \frac{|W_0^M|}{|W_0^G|} I_M^G, \quad I^G \in \{I_{\text{geom}}, I_{\text{spec}}\}$$

- M_0: a fixed minimal Levi of G,
- W_0^M: the Weyl group relative to $M_0 \subset M$,
- I_M^G: invariant distribution with an expansion indexed by classes γ (resp. irreps π) in M.

Based by truncation + a plethora of other tools.

Dramatis personae

Let M be a Levi of G.

Terms of local nature: Let f be a test function on $G(\mathbb{A})$.

- $I_M^G(\gamma, f)$: the INVARIANT VERSION of WEIGHTED orbital integrals, where γ: conjugacy classes in M,
- $I_M^G(\pi, f)$: the INVARIANT VERSION of WEIGHTED characters, where π: unitary representation of M.

When $G = M$, we recover the usual orbital integrals and characters.

Terms of global nature: the coefficients

- expressing $I_{M,\mathrm{geom}}^G(f)$ in terms of $I_M^G(\gamma, f)$,
- expressing $I_{M,\mathrm{spec}}^G(f)$ in terms of $I_M^G(\pi, f)$.

Ultimately, we want to understand the distributions

$$\boxed{I^G_{\mathrm{spec}}, \quad I^G_{\mathrm{disc}}, \quad I^G_{\mathrm{disc},\nu}, \quad I^G_{\mathrm{disc},\nu,c^V}}$$

on $G(\mathbb{A})$, where we specified

- ν: infinitesimal character,
- c^V: Satake parameter off V, where V is a large finite set of places.

$$I^G_{\mathrm{disc}} = \underbrace{\mathrm{tr}\left(L^2_{\mathrm{disc}}\right)}_{=\sum_\pi m(\pi)\,\mathrm{tr}(\pi)} + \text{“shadows” from Levi.}$$

The “shadows” are closely related to some key ingredients in Arthur’s conjectures — local and global intertwining relations, or the structure of parabolically induced packets.

Known applications

They usually require a stable trace formula and its twisted analogue (Arthur, Mœglin–Waldspurger, ...), based on (twisted) *Endoscopy* by Langlands–Shelstad–Kottwitz.

$$I^G(f) = \sum_{\substack{\mathbf{G}' \\ \text{ell. endo. data}}} \iota(G, \mathbf{G}') S^{G'}(f'),$$

- I^G: the invariant distribution to be stabilized;
- $S^{G'}$: stable counterparts on the endoscopic group G' (quasisplit), defined recursively;
- $\iota(G, \mathbf{G}') \in \mathbb{Q}_{>0}$: explicit coefficients;
- $f \mapsto f'$: transfer of test functions from G to G' (of a local nature).

We now move to the metaplectic case.

The metaplectic cover

Let $\mathrm{Sp}(2n) \subset \mathrm{GL}(2n)$ be the symplectic group. Let $\mu_m = \{z \in \mathbb{C}^\times : z^m = 1\}$. The global METAPLECTIC COVERING is a central extension of locally compact groups

$$1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(2n, \mathbb{A}) \to \mathrm{Sp}(2n, \mathbb{A}) \to 1,$$

$$\mathrm{Mp}(2n, \mathbb{A}) := \widetilde{\mathrm{Sp}}(2n, \mathbb{A})_{\mathrm{der}} : \text{twofold cover.}$$

- There is a canonical splitting over $\mathrm{Sp}(2n, F)$.
- It depends on a symplectic space $(W, \langle \cdot | \cdot \rangle)$ and an additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$.
- It is the restricted product of local coverings $1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(2n)_v \to \mathrm{Sp}(2n, F_v) \to 1$, modulo $\{(z_v)_v \in \bigoplus_v \mu_8 : \prod_v z_v = 1\}$.

1. We are interested in studying genuine representations and automorphic forms of $\widetilde{\mathrm{Sp}}(2n)$, i.e. on which μ_8 acts by $z \mapsto z \cdot \mathrm{id}$.
2. The genuine representation theory of $\widetilde{\mathrm{Sp}}(2n)$ (both local and global) are largely elucidated by Gan–Savin, Gan–Ichino, using Θ.
3. A model for *Langlands' program for covering group* (Weissman, Gan, Gao, ...)
4. Other Brylinski–Deligne coverings occurring naturally:
   - coverings of $\mathrm{GL}(n)$ (Kazhdan–Patterson),
   - higher coverings of symplectic groups (Friedberg, Ginzburg *et al.*),
   - ......

Key feature of $\widetilde{\mathrm{Sp}}(2n)$: two elements $\tilde{\delta}, \tilde{\delta}'$ commute in $\widetilde{\mathrm{Sp}}(2n)_v$ iff their images $\delta, \delta' \in \mathrm{Sp}(2n, F_v)$ commute.

Invariant trace formula for coverings

Most results in harmonic analysis extend to coverings. The invariant trace formula à la Arthur ▸ Cf. linear version

$$I^{\tilde{G}} = \sum_{M} \frac{|W_0^M|}{|W_0^G|} I^{\tilde{G}}_{\tilde{M}}$$

is known under the following technical assumptions.

- *Satake isomorphism* at the unramified places (✓ for BD-coverings),
- *Trace Paley–Wiener theorem* for K-finite functions at Archimedean places (✓ for $\widetilde{\mathrm{Sp}}(2n)$ and its Levi).

What remains is a stabilization à la Arthur. This requires a theory of endoscopy for coverings.

Endoscopy for $\widetilde{\mathrm{Sp}}(2n)$

Let $\tilde{G} = \widetilde{\mathrm{Sp}}(2n)$, $G = \mathrm{Sp}(2n)$. In both local and global cases:

- Dual group: $\tilde{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$ with trivial Galois action.
- Elliptic endoscopic data $\mathbf{G}^! \leftrightarrow$ pairs $(n', n'') \in \mathbb{Z}^2_{\geq 0}$ such that $n' + n'' = n$. NO SYMMETRY HERE!
- Endoscopic group associated with $\mathbf{G}^!$: $G^! = \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$, split.
- Can define
  - a correspondence of stable semisimple conjugacy classes,
  - the factors $\iota(\tilde{G}, \mathbf{G}^!)$ as before,
  - transfer factors Δ.

Note. Over every Levi $\prod_i \mathrm{GL}(n_i) \times \mathrm{Sp}(2m)$ of G, the 8-fold covering splits canonically into $\prod_i \mathrm{GL}(n_i, F) \times \widetilde{\mathrm{Sp}}(2m)$.

The notion of transfer

To study genuine representations, we consider anti-genuine test functions[1] on $\tilde{G}$ (local).
For each $\mathbf{G}^!$ we have the transfer of test functions

$$C^\infty_{c,\mathrm{anti-gen.}}(\tilde{G}) \dashrightarrow C^\infty_c(G^!)$$

$$f \longmapsto f^!$$

whose orbital integrals are matching in the sense that

$$\underbrace{S_{G^!}(\delta, f^!)}_{\text{stable orbital integral}} = \sum_{\gamma \leftrightarrow \delta} \Delta(\delta, \tilde{\gamma}) \underbrace{I_{\tilde{G}}(\tilde{\gamma}, f)}_{\text{orbital integral}}, \quad \begin{array}{l} \delta : \text{st. conj. class in } G^!(F) \\ \gamma : \text{conj. class in } G(F) \end{array}$$

where $\tilde{\gamma} \mapsto \gamma$ is arbitrary. Thus Δ plays the role of “kernel”.

[1] i.e. $f(z\tilde{x}) = z^{-1} f(\tilde{x})$ for all $z \in \mu_8$.

Known results

1. Existence of transfer is known (descent + results of Ngo *et al.* on Lie algebras).
2. Dual of transfer: stable character $\mapsto$ virtual character.
3. In the unramified local case, we have:
   - Fundamental Lemma for units.
   - Fundamental Lemma for spherical Hecke algebras (Caihua Luo) $\rightsquigarrow$ transfer of Satake parameters.
   - Weighted Fundamental Lemma.
4. The elliptic semisimple part in $I^{\tilde{G}}_{\mathrm{geom}}$ has been stabilized.

These results concern only the $M = G$ part in the trace formula!

The stabilized trace formula

Main Theorem

Consider the global covering $\tilde{G} \twoheadrightarrow G(\mathbb{A})$. For every $f = \prod_v f_v \in C^\infty_{c,\text{anti-gen.}}(\tilde{G})$, we have

$$I^{\tilde{G}}(f) = \sum_{\mathbf{G}^!:\text{ell. endo. data}} \iota(\tilde{G}, \mathbf{G}^!) S^{G^!}(f^!),$$

where

- $f^! = \prod_v f_v^!$ is a transfer of f to $G^!(\mathbb{A})$,
- $S^{G^!}$ is the stable distribution obtained in Arthur's stabilization.

▸ Cf. linear version

The spectral expansion of $S^{G^!}$ is given by the *stable multiplicity formula* of Arthur for split odd SO.

Potential applications

The formula

$$I^{\tilde{G}}_{\mathrm{disc}}(f) = \sum_{\mathbf{G}^!} \iota(\tilde{G}, \mathbf{G}^!) S^{G^!}_{\mathrm{disc}}(f^!).$$

should yield

- information about $L^2_{\mathrm{disc,genuine}}(G(F)\backslash\tilde{G})$
- LLC for local $\widetilde{\mathrm{Sp}}(2n)$ + endoscopic character relations.

- The LLC is known via Θ (Gan–Savin); its compatibility with endoscopic character relations is verified by Caihua Luo.
- Using Θ, Gan and Ichino already obtained a multiplicity formula for the *tempered automorphic spectrum*, fitting into Arthur's conjecture.[2]
- If successful, the stable trace formula should be able to tackle the general non-tempered case.

In Arthur's approach, the local and global intertwining relations will play a crucial role. Cf. Ishimoto's work for tempered case.

ϵ-factor**s**

[2] They also obtain the non-tempered case for $\widetilde{\mathrm{Sp}}(4)$.

Terms in the invariant trace formula

The terms in $I^{\tilde{G}}_{\mathrm{geom}} = I^{\tilde{G}}_{\mathrm{spec}}$ are catalogued as follows.

Global distributions	Global "coefficients"	Semi-local distributions
$I^{\tilde{G}}_{\mathrm{geom}}$	$A^{\tilde{G}}(V, \mathscr{O})$	$I^{\tilde{G}_V}_{\tilde{M}_V}(\tilde{\gamma}, f)$
$I^{\tilde{G}}_{\mathrm{spec}}$	$I^{\tilde{G}}_{\mathrm{disc},t}$	$I^{\tilde{G}_V}_{\tilde{M}_V}(\pi_V, \nu, X, f)$

- V: finite set of places containing ramified ones.
- The semi-local distributions depend on a Levi $M \subset G$ and admit splitting formulas into local avatars.

The superscripts $\tilde{G}$ are often omitted.

Road map

Each distribution/coefficient has an ENDOSCOPIC COUNTERPART (with superscript $\mathscr{E}$).

Idea: TERM-BY-TERM STABILIZATION

$$I = I^{\mathscr{E}}, \quad \underbrace{A = A^{\mathscr{E}}}_{\text{global}}, \quad \underbrace{I^{\tilde{G}}_{\tilde{M}} = I^{\tilde{G},\mathscr{E}}_{\tilde{M}}}_{\text{(semi-)local}}.$$

- By induction, in the (semi-)local case we assume

$$I^S_L = I^{S,\mathscr{E}}_L$$

when $M \subset L \subset S \subset G$ are Levi, $M \neq L$ or $S \neq G$.
- Bootstrapping from the known case $M = G$. ▸ Preview

1. Properties of I itself are often proved in the same way as the uncovered case — they are *of an analytic nature* ✓.
2. The STABLE COUNTERPART $S = S^{G^!}$ lives on endoscopic groups $G^!$ — ✓. We even have Arthur's endoscopic classification for $G^!$.
3. The endoscopic counterpart $I^{\mathscr{E}}$ is made from various $S^{G^!}$ via *transfer*. ▸ An example
   This part requires new combinatorial/cohomological arguments.

Ideally, the first step would be on the geometric side.

The global geometric statement

Consider the metaplectic covering $1 \to \mu_8 \to \tilde{G} \to G(\mathbb{A}_F) \to 1$.

- $\mathscr{O}$: semisimple stable class in $G(F)$, determining a finite set of places $S(\mathscr{O}) \supset \{v : v \mid \infty\}$.
- $A^{\tilde{G}}(S, \mathscr{O})_{\mathrm{ell}}$: formal linear combination of orbits in $\tilde{G}_S$. It is the building block in the expansion of $I^{\tilde{G}}_{\tilde{G},\mathrm{geom}}$ indexed by $\mathscr{O}$, and $S \supset S(\mathscr{O})$.
- $A^{\tilde{G},\mathscr{E}}(S, \mathscr{O})_{\mathrm{ell}}$: the endoscopic analogue.

Global Geometric Theorem

For each elliptic semisimple stable class $\mathscr{O}$ in $G(F)$,

$$A^{\tilde{G}}(S, \mathscr{O})_{\mathrm{ell}} = A^{\tilde{G},\mathscr{E}}(S, \mathscr{O})_{\mathrm{ell}}.$$

This stabilizes the global COEFFICIENTS in I_{geom}.

The local geometric statement

Consider the local $1 \to \mu_8 \to \tilde{G} \to G(F) \to 1$.

Local Geometric Theorem

Let $M \subset G$ be a Levi, $\tilde{\gamma}$ an $M(F)$-conjugacy class in $\tilde{M}$ (more generally, a “geometric” invariant distribution), then

$$I_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, f) = I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\tilde{\gamma}, f)$$

for all anti-genuine f.

Here, $I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\tilde{\gamma}, \cdot)$ is the endoscopic avatar of the geometric distribution $I_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, \cdot)$ in the invariant trace formula for $\tilde{G}$.

Weighted Fundamental Lemma (proven)

The unramified version of the above:

$$r_{\tilde{M}}(\tilde{\gamma}, K) = r_{\tilde{M}}^{\mathscr{E}}(\tilde{\gamma}).$$

Specifically,

$$I_{\tilde{M}}^{\tilde{G},\mathscr{E}}(\mathbf{M}^!, \delta, f) = \sum_s i_{M^!}(\tilde{G}, G^![s]) S_{M^!}^{G^![s]}\left(\delta[s], B, f^{G^![s]}\right),$$

where s indexes diagrams

$$\begin{array}{ccc} G^![s] & \overset{\text{ell.}}{\underset{\text{endo.}}{\leftarrow\!\!-\!-\!\!\rightarrow}} & \tilde{G} \\ \text{Levi}\uparrow & & \uparrow\text{Levi} \\ M^! & \overset{\text{ell.}}{\underset{\text{endo.}}{\leftarrow\!\!-\!-\!\!\rightarrow}} & \tilde{M} \end{array}$$

- δ is a stable geometric distribution $M^!(F)$,
- $i_{M^!}(\tilde{G}, G^![s])$ are explicit constants defined by dual groups,
- $S_{M^!}^{G^![s]}(\cdots)$ are the stable distributions from Arthur,
- $\delta \mapsto \delta[s]$ is a twist by some central element $z[s] \in M^!(F)$. A metaplectic feature!

B-functions

The B above prescribes an adjustment of root-lengths in $M^!_\delta$ and $G[s]^!_{\delta[s]}$. Here: type $\mathrm{B}_m \leftrightarrow \mathrm{C}_m$.

- It affects the definition of weighted orbital integrals (Mœglin–Waldspurger).
- It fades away in the global setting.

One shows that $I^{\tilde{G},\mathscr{E}}_{\tilde{M}}(\mathbf{M}^!, \delta, f)$ depends only on the transfer of δ to $\tilde{M}$. This defines $I^{\tilde{G},\mathscr{E}}_{\tilde{M}}(\tilde{\gamma}, f)$.
When $G = M$ and γ is regular, we recover the *transfer of orbital integrals*.

Strategy

1. The Global Geometric Theorem has a RELATIVELY SHORT proof. Ingredients:
   - Descent: use known results concerning various $A^{G_\gamma}_{\mathrm{unip}}(\cdots)$ (Arthur, Moeglin–Waldspurger).
   - Play with Δ.
   - Manipulation of non-abelian Galois cohomologies.
2. The Local Geometric Theorem requires more efforts.
   - Local trace formula and its stabilization (inductive assumption).
   - Stabilization of the spectral side of the global trace formula (special cases).
   - Local–global argument. ▸ Preview

For this purpose, we need to use the SPECTRAL SIDE.

Reduction of the local geometric theorem to G-regular case

Idea: Yoga of germs.

- F non-Archimedean: descent + Shalika germs + known results from Arthur and Mœglin–Waldspurger (nonstandard endoscopy).
- F Archimedean: more difficult — a subtle analysis of the maps ρ_J, σ_J (“germs”) defined à la Mœglin–Waldspurger[3].

In our case, coverings of the form

$$1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(2a) \overset{\mu_8}{\times} \widetilde{\mathrm{Sp}}(2b) \to \mathrm{Sp}(2a, F) \times \mathrm{Sp}(2b, F) \to 1$$

will intervene. Not too far away from $\widetilde{\mathrm{Sp}}$!

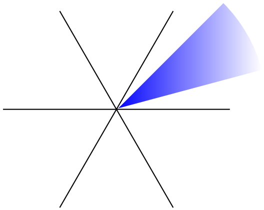
[3] $J \approx$ subsets of roots restricted to A_M

As germs in $a \in A_M(F)$, $a \to 1$:.

$$I_{\tilde{M}}\left(a\tilde{\gamma}, f\right) \sim \sum_{L \in \mathscr{L}(M)} \sum_{J \in \mathscr{J}_M^L} I_{\tilde{L}}\left(\rho_J^{\tilde{L}}(\tilde{\gamma}, a)^{\tilde{L}}, f\right)$$

where $u \sim u'$ means: there exists $r > 0$ such that $|u'(a) - u(a)| \ll \|\log(a)\|^r$ as

$\mathfrak{a}_M \ni \log a \;\swarrow$ in



The arguments can be made uniform for all F.

Cancellation of singularities

Encapsulate the obstruction to the G-regular local geometric theorem into an orbital integral.

Theorem

There exists $\epsilon_{\tilde{M}}(\cdot)$, mapping f to a cuspidal anti-genuine test function on $\tilde{M}$, whose usual orbital integral satisfies

$$I^{\tilde{M}}(\tilde{\gamma}, \epsilon_{\tilde{M}}(f)) = I^{\tilde{G},\mathscr{E}}_{\tilde{M}}(\tilde{\gamma}, f) - I^{\tilde{G}}_{\tilde{M}}(\tilde{\gamma}, f).$$

- This requires new "compactly-supported" distributions ${}^{c}I_{\tilde{M}}(\tilde{\gamma}, \cdot)$ and their stabilization.
- Also have to stabilize certain maps

$$ {}^{c}\theta_{\tilde{M}} : \text{test fcn on } \tilde{G} \to \text{test fcn on } \tilde{M}$$

  relating $I_{\tilde{M}}$ and ${}^{c}I_{\tilde{M}}$.

Concerning the construction of $\epsilon_{\tilde{M}}(\cdot)$:

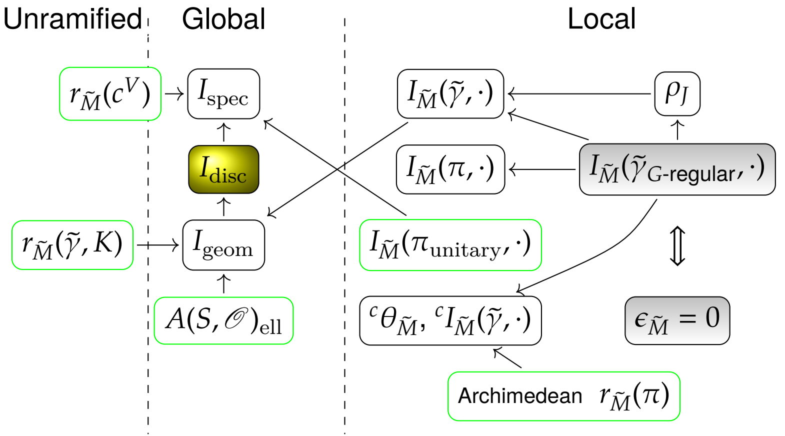
- For Archimedean F, we have to normalize the intertwining operators BY HAND, and stabilize some factors

$$r_{\tilde{M}}(\pi), \quad \pi : \text{unitary genuine irrep of } \tilde{M}$$

  arising from a (G, M)-family associated with normalizing factors —
- We also need to stabilize the *differential equations* and *jump conditions* satisfied weighted orbital integrals.

A similar scenario in the global setting: Stabilize $r^{\tilde{G}}_{\tilde{M}}(c^V)$ arising from unramified normalizing factors, where

$$V : \text{ large finite set of places,}$$
$$c^V : \text{ quasi-automorphic Satake parameter off } V.$$



- $\boxed{A}$ means A can be stabilized directly.
- $A \to B$ means the stabilization of A is NEEDED to stabilize B.

The final touch

Take an elliptic endoscopic datum $\mathbf{M}^!$ for $\tilde{M}$. Define

$$\epsilon_{\tilde{M}}^{\mathbf{M}^!}(f)(\delta) := \sum_{\gamma} \Delta(\delta, \tilde{\gamma}) \underbrace{I^{\tilde{M}}(\tilde{\gamma}, \epsilon_{\tilde{M}}(f))}_{\text{usual orbital integral}}$$

$$= \left(\text{transfer of } \epsilon_{\tilde{M}}(f)\right)(\delta)$$

for all stable regular semisimple class δ in $M^!(F)$.
Here $\cdots(\delta)$ means taking stable orbital integral along δ.

Goal

Show that $\epsilon_{\tilde{M}}^{\mathbf{M}^!}(f) = 0$ for all $\mathbf{M}^!$.

Strategy: Show it is both REAL and IMAGINARY-valued.

Let $f_{\tilde{M}}^{\mathbf{M}^!}$ be the transfer of the parabolic descent $f_{\tilde{M}}$ of f to $M^!$.

Key geometric hypothesis

There is a smooth function $\epsilon(\mathbf{M}^!, \cdot)$ on $M^!_{M\text{-reg}}(F)$ such that

$$\epsilon_{\tilde{M}}^{\mathbf{M}^!}(f)(\delta) = \epsilon(\mathbf{M}^!, \delta) f_{\tilde{M}}^{\mathbf{M}^!}(\delta) \quad \text{for all } f, \delta.$$

This is established by a local–global argument, by stabilizing a not-so-simple global trace formula and using its SPECTRAL SIDE.

Imaginary Lemma

We have $\epsilon(\mathbf{M}^!, \delta) + \overline{\epsilon(\mathbf{M}^!, \delta)} = 0$ for all $\mathbf{M}^!$ and δ.

Proof is based on the local trace formula:

- Use a pair of test functions $(\overline{f_1}, f_2)$ where $f_i \in C_c^\infty(\tilde{G})$ is anti-genuine, $i = 1, 2$.
- Hence $\overline{f_1}$ is anti-genuine over the *antipodal* covering $\tilde{G}^\dagger$, i.e. $\tilde{G}^\dagger = \tilde{G}$ but $\mu_8 \to \tilde{G}^\dagger$ is modified by $z \mapsto z^{-1}$.
- The correct way of looking at the local trace formula is to consider the pair $(\tilde{G}^\dagger, \tilde{G})$.
- If $\tilde{G} = \widetilde{\mathrm{Sp}}(W, \langle\cdot|\cdot\rangle)$ then $\tilde{G}^\dagger$ with $\tilde{G}_- := \widetilde{\mathrm{Sp}}(W, -\langle\cdot|\cdot\rangle)$.

Antipodal vs. transfer

Flipping $\langle\cdot|\cdot\rangle$ does not alter endoscopic data/correspondence of classes, whilst it takes Δ to $\overline{\Delta}$.

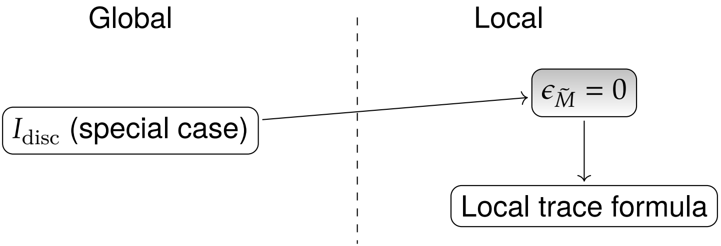
Proof: Maktouf's character formula for ω_ψ (Maslov indices, etc.)

Real Lemma

We have $\epsilon(\mathrm{M}^!, \delta) = \overline{\epsilon(\mathrm{M}^!, \delta)}$ for all $\mathrm{M}^!$ and δ.

- It boils down to showing that endoscopic transfer is "isomorphic to its complex conjugate".
- This we can achieve by the MVW-involution $\tilde{\mathrm{G}} \xrightarrow{\sim} \tilde{\mathrm{G}}_-$, realized by $\mathrm{Ad}(g)$ with $g \in \mathrm{GSp}(W)$ with similitude -1.

In the uncovered case and its twisted analogue, the *Chevalley involution* is used by Arthur and Mœglin–Waldspurger.



▸ Cf. an earlier diagram

☞ Both “special case” and “imaginary lemma” involve a famous method (from Jacquet–Langlands?) — if there is an equality between continuous and discrete spectral expansions, then both sides $= 0$.

Thanks for your attention.

Image taken from **Bing**
Last update: October 24, 2021