# On the A-packets for genuine representations of Mp(2n)

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## Outline

Genuine representations of metaplectic groups

LLC of Gan–Savin

Automorphic set-up

Desiderata: local and global

Results of Gan–Ichino

Strategy à la Arthur

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Hecke algebra correspondences

References

## Local metaplectic covering

- $\mu_m := \mu_m(\mathbb{C})$ , all rep are over  $\mathbb{C}$ .
- *F*: local field, char(F) = 0; let  $\mathcal{L}_F$  denote its Weil–Deligne group.
- W: symplectic F-vector space, dim W = 2n.
- Sp(W) = Sp(W, F) (or Sp(2n, F)): the symplectic group.

## Definition

If  $F \neq \mathbb{C}$ , the metaplectic group is THE non-trivial central extension of locally compact groups

$$1 \rightarrow \mu_2 \rightarrow \mathsf{Mp}(W) \rightarrow \mathsf{Sp}(W) \rightarrow 1.$$

If  $F = \mathbb{C}$  we put  $Mp(W) = \mu_2 \times Sp(W)$ .

It is customary to write Mp(2n) or Mp(2n, F).

## Genuine representations

Fixing an additive character  $\psi$  of F and the symplectic form  $\langle \cdot | \cdot \rangle$  on W, one can describe Mp(W) by explicit 2-cocycles (Rao, Lion–Perrin).

*Representations* of Mp(*W*): HC-modules or Casselman–Wallach representations if  $F \supset \mathbb{R}$ ; smooth if  $F \supset \mathbb{Q}_p$ .

## Definition

A representation  $(\pi, V_{\pi})$  of Mp(W) is genuine if  $\pi(z) = z \cdot id_{V_{\pi}}$  for all  $z \in \mu_2$ .

For instance, the Weil/oscillator representation  $\omega_{\psi} = \omega_{\psi}^+ \oplus \omega_{\psi}^-$  of Mp(W) is genuine. They depend on  $\psi \circ \langle \cdot | \cdot \rangle$ .

## Goal

Understand the genuine representation theory of Mp(W).

## The L-group

Question: Langlands program for genuine representations of Mp(W)?

Fix  $\psi \circ \langle \cdot | \cdot \rangle$ . There are strong evidences (from  $\Theta$ -correspondence, geometric Satake, etc.) for the

## Definition

The L-group of Mp(W) is  $Sp(2n, \mathbb{C}) \times Weil_F$ , i.e. same as the L-group of the split SO(2n + 1).

This is also compatible with Weissman's definition of L-groups for coverings.

- L-parameters for Mp(W) = symplectic representations  $\phi = \bigoplus_{i \in I} m_i \phi_i$  of  $\mathcal{L}_F$ , where  $m_i \ge 1$ , the  $\phi_i$ 's are distinct simple representations of  $\mathcal{L}_F$ , and  $\sum_i m_i \dim \phi_i = 2n$ .
- A-parameters for Mp(W) = symplectic representations  $\psi = \bigoplus_{i \in I} m_i \psi_i$  of dimension 2n of  $\mathcal{L}_F \times SL(2, \mathbb{C})$  as above; the  $\mathcal{L}_F$  factor of each  $\psi_i$  is bounded.
- $S_{\Phi}$  = centralizer of  $\phi$  in Sp(2*n*,  $\mathbb{C}$ ) (same for  $S_{\psi}$ ).

• 
$$S_{\Phi} = \pi_{o}(S_{\Phi})$$
 (same for  $S_{\Psi}$ ).

One can describe  $S_{\Phi}$  and  $S_{\Phi}$  explicitly;  $S_{\Phi}$  is finite abelian, and:

$$\mathcal{S}_{\Phi}^{\vee} = \boldsymbol{\mu}_{2}^{I^{+}}, \quad I^{+} := \{i \in I : \phi_{i} \text{ symplectic.}\}$$

Same for  $S_{\psi}$ ,  $S_{\psi}$  and  $S_{\psi}^{\vee}$ .

## Local Langlands correspondences

Let  $\Phi_{\text{symp}}(2n)$  be the set of equivalence classes of L-parameters for Mp(W) (= those for SO(2n + 1)). The following is due to Adams–Barbasch ( $F \supset \mathbb{R}$ ) and Gan–Savin ( $F \supset \mathbb{Q}_p$ ).

#### Theorem (LLC)

There is a decomposition

$$\operatorname{Irr}_{\operatorname{gen}}(\operatorname{Mp}(W)) = \bigsqcup_{\Phi \in \Phi_{\operatorname{sympl}(2n)}} \Pi_{\Phi},$$

together with bijections  $\Pi_{\Phi} \leftrightarrow \mathbb{S}_{\Phi}^{\vee}$  + various properties, eg.

- $\pi \in \Pi_{\Phi}$  is tempered (resp. discrete series)  $\iff \phi$  is bounded (resp. does not factor through proper Levi);
- LLC reduces to tempered/bounded case via Langlands quotients;
- if  $\varphi$  is bounded, then  $\textbf{i} \in \mathbb{S}_{\Phi}^{\vee}$  corresponds to generic representation.

- The LLC depends on  $\psi$  and the symplectic form  $\langle\cdot|\cdot\rangle$  on W.
- It is proved by reduction to the LLC of SO( $V^{\pm}$ ) (Arthur, Ishimoto) where  $V^{\pm}$  is the quadratic vector space with
  - $\circ$  dimension 2n + 1,
  - discriminant 1,
  - $\circ~$  Hasse invariant  $\pm 1$  ,

via  $\Theta$ -correspondence for the reductive dual pair (Sp(W), O(V<sup>±</sup>)).

- Note:  $\mathbb{S}_{\Phi}^{\vee}$  is also in bijection with the packet  $\Pi_{\Phi}^{Vogan}$  for  $SO(V^{\pm})$ .
- There is a more direct proof for  $F = \mathbb{C}$ .

## The endoscopic viewpoint (Adams, Renard, L.)

The set of elliptic endoscopic data of Mp(W) is defined as

$$\mathcal{E}_{\mathsf{ell}}(\mathsf{Mp}(W)) := \{s \in \mathsf{Sp}(2n, \mathbb{C}) : s^2 = 1\} / \mathsf{conj}.$$
$$= \{(\underbrace{n'}_{+}, \underbrace{n''}_{-}) \in \mathbb{Z}^2_{\geq 0} : n' + n'' = n\}.$$

The endoscopic group is  $SO(2n' + 1) \times SO(2n'' + 1)$ .

Similar to elliptic endoscopic data for SO(2*n* + 1), but without symmetry  $(n', n'') \leftrightarrow (n'', n')$ .

#### Known results

- Transfer of orbital integrals (Renard for  $F = \mathbb{R}$ , L. '11 for  $F \supset \mathbb{Q}_p$ )
- Fundamental lemma for units, including the weighted case (L. '11)
- Fundamental lemma for spherical Hecke algebra (C. Luo '18).

## Let $\mathbf{G}^! \in \mathcal{E}_{ell}(Mp(W))$ with endoscopic group $G^!$ . The transfer of orbital integrals dualizes to a map

 $\check{\Upsilon}_{\mathbf{G}^{!},\mathsf{Mp}(W)}: \{ \text{st. dist. on } G^{!}(F) \} \rightarrow \{ \text{genuine dist. on } \mathsf{Mp}(W) \}$ sending stable virtual characters to genuine virtual characters.

- For every BOUNDED L-parameter  $\phi^!$  for  $G^!$ , we have a stable tempered distribution  $S\Theta_{\phi^!}^{G^!}$ .
- $(G^!)^{\vee} = \operatorname{Sp}(2n', \mathbb{C}) \times \operatorname{Sp}(2n'', \mathbb{C}) \hookrightarrow \operatorname{Sp}(2n, \mathbb{C})$  up to conjugacy, hence  $\phi^!$  maps to a bounded  $\phi \in \Phi_{\operatorname{symp}}(2n)$ .

#### Endoscopic character relations (ECR) — C. Luo

Let  $\phi \in \Phi_{symp}(2n)$  be bounded. The L-packet  $\Pi_{\phi}$  for Mp(W) can be characterized in terms of  $\check{T}_{\mathbf{G}^{!},Mp(W)}\left(S\Theta_{\Phi^{!}}^{G^{!}}\right)$  for various  $(\mathbf{G}^{!},\phi^{!})$  such that  $\phi^{!} \mapsto \phi$ , and some  $\epsilon$ -factors.

The precise formulation will be given later on.

## The adélic covering

Let F be a number field, and W a symplectic F-vector space of dimension 2n. The metaplectic group is the non-trivial central extension

$$1 \rightarrow \boldsymbol{\mu}_2 \rightarrow \mathsf{Mp}(W, \mathbb{A}_F) \rightarrow \mathbf{Sp}(W, \mathbb{A}_F) \rightarrow 1.$$

- It splits uniquely over Sp(W, F), hence it makes sense to study
   *genuine automorphic forms* on Mp(W, A<sub>F</sub>), eg. Siegel modular forms of <sup>1</sup>/<sub>2</sub> + Z-weights,
  - genuine *L*<sup>2</sup>-automorphic spectrum.
- It is the quotient of Π<sup>'</sup><sub>ν</sub> Mp(W<sub>ν</sub>) by {(z<sub>ν</sub>)<sub>ν</sub> ∈ ⊕<sub>ν</sub> μ<sub>2</sub> : Π<sub>ν</sub> z<sub>ν</sub> = 1}, hence irreducible admissibles of Mp(W, A<sub>F</sub>) decompose into ⊗<sup>'</sup><sub>ν</sub> π<sub>ν</sub>.

Here we use the fact that  $Mp(W_{\nu})$  splits over  $Sp(W_{\nu}, \mathcal{O}_{\nu})$  with commutative Hecke algebras, for almost all  $\nu$ .

## Local desiderata for Arthur packets

Let *F* be local, dim<sub>*F*</sub> W = 2n and  $\psi$  is fixed. For the study of

- 1. unitary duals,
- 2. Gelfand–Kirillov dimensions, or
- 3. global *L*<sup>2</sup>-automorphic spectrum,

the LLC is not enough: one has to go beyond tempered L-packets and study Arthur packets. My main motivation is 3.

Recall: A-parameters for Mp(W) are symplectic representations

$$\psi = \bigoplus_{i \in I} m_i \phi_i \boxtimes r(b_i) : \mathcal{L}_F \times SL(2, \mathbb{C}) \to GL(2n, \mathbb{C}),$$

where  $r(b_i) := \text{Sym}^{b_i-1}(\text{std})$  and  $\phi_i$  is bounded. Let  $\Psi_{\text{symp}}(2n) = \{\text{such } \psi\}$ . Define  $\Psi_{\text{symp}}^+(2n)$  by dropping boundedness.

## Characterization via ECR

Given a pair ( $\mathbf{G}^{!}, \psi^{!}$ ) where  $\mathbf{G}^{!} \in \mathcal{E}_{\mathsf{ell}}(\mathsf{Mp}(W))$  and  $\psi^{!} \in \Psi^{+}(G^{!})$ , we obtain  $(\psi, s)$  where  $\psi^{!} \mapsto \psi$  and  $s \in S_{\psi, 2\text{-tors}}/\mathsf{conj}$  corresponds to  $\mathbf{G}^{!}$ . This is actually a bijection.

Given  $(\psi, s)$ , Arthur's theory for  $G^!$  provides stable virtual characters  $S\Theta_{\psi^!}^{G^!}$  on  $G^!(F)$ . Consider the (-1)-eigenspace of std  $\circ \psi$  under *s* and set

$$\epsilon(\psi^{s=-1}) := \epsilon\left(\frac{1}{2}, \psi^{s=-1}\Big|_{\mathcal{L}_F}, \psi\right).$$

#### **Definition-Lemma**

The following depends only on  $\psi$  and the image *x* of *s* in  $S_{\psi}$ .

$$T_{\psi,s} := \boldsymbol{\epsilon} \left( \boldsymbol{\psi}^{s=-1} \right) \cdot \check{\mathfrak{T}}_{\mathbf{G}^{!},\mathsf{Mp}(W)} \left( S \Theta_{\boldsymbol{\psi}^{!}}^{G^{!}} \right).$$

Every  $x \in S_{\psi}$  arises from some  $s \in S_{\psi,2-\text{tors}}$ , hence we may write  $T_{\psi,x} = T_{\psi,s}$ , and consider the Fourier expansion of  $x \mapsto T_{\psi,x}$  or its translates.

#### Main local Theorem (L.)

Given  $\psi \in \Psi^+_{\text{symp}}(2n)$ , set  $s_{\psi} := \psi(1, \binom{-1}{-1}) \in S_{\psi}$  and let  $x_{\psi}$  be its image in  $S_{\psi}$ . Then

$$\pi_{\psi,\chi} := |\mathcal{S}_{\psi}|^{-1} \sum_{x \in \mathcal{S}_{\psi}} \chi(x_{\psi}x) T_{\psi,x}$$

is a  $\mathbb{Z}_{\geq 0}$ -linear combination (possibly zero) of genuine irreducible characters of Mp(W), for all  $\chi \in \mathbb{S}_{\psi}^{\vee}$ . If  $\psi \in \Psi_{symp}(2n)$ , then these irreducible characters arise from unitary representations.

We also call it the endoscopic character relation (ECR) associated with  $\psi$ .

#### **Definition of A-packets**

Collecting the constituents of  $\pi_{\psi,\chi}$  for various  $\chi$ , we obtain the A-packet  $\Pi_{\psi}$  as a multi-set of genuine irreducibles. Rigorously,  $\Pi_{\psi}$  is a finite set equipped with two maps

$$\operatorname{Irr}_{\operatorname{gen}}(\operatorname{Mp}(W)) \leftarrow \Pi_{\psi} \to \mathbb{S}_{\psi}^{\vee}.$$

- The structure above is characterized completely by ECR.
- The e-factor in  $T_{\psi,s}$  is a metaplectic feature: it does not appear in Arthur's original version.
- If  $\psi$  is a bounded L-parameter (i.e. all  $b_i = 1$ ), then we recover the L-packet and Luo's ECR for the tempered LLC for Mp(W).

Furthermore,  $\pi_{\psi,\chi}$  has the following properties established in [L.].

- 1. Reduction to good parity case (i.e. each simple summand in  $\psi$  is symplectic) via full parabolic induction.
- 2. Infinitesimal characters expressed in terms of  $\psi$  when  $F \supset \mathbb{R}$ .
- 3. Assume the covering is unramified. If  $\psi$  is trivial on  $I_F \times SL(2, \mathbb{C}) \subset \mathcal{L}_F$ , then  $\Pi_{\psi}$  has a unique spherical member, which is multiplicity-free and parametrized by  $\chi = \mathbf{1}$ . If  $\psi$  is not unramified then  $\Pi_{\psi}$  has no spherical members.
- 4. Central characters expressed in terms of  $\varepsilon\text{-factors}$  and  $(\psi,\chi).$
- 5.  $\Pi_{\phi_{\psi}}$  embeds canonically into  $\Pi_{\psi}^{\text{mult}=1}$ , where for all  $w \in \mathcal{L}_F$ ,  $\phi_{+}(w) = \psi\left(w \left(\frac{|w|^{1/2}}{2}\right)\right)$  (L-parameter)

$$\Phi_{\Psi}(w) = \Psi\left(w, \left( \begin{smallmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{smallmatrix} 
ight) 
ight)$$
 (L-parameter).

- 6. The effect of variation of  $\psi$  can be explicitly described, generalizing the recipe for L-packets due to Gan–Savin.
- 7. Normalization of int. op. via A-parameters.

## Global desiderata

Let F be a number field. As before, dim<sub>F</sub> W = 2n and  $\psi = \bigotimes_{\nu} \psi_{\nu} : F \setminus \mathbb{A}_F \to \mathbb{C}^{\times}$  is fixed. Put  $L^2_{\text{gen,disc}} := L^2_{\text{genuine,discrete}}(\mathbf{Sp}(W, F) \setminus \mathsf{Mp}(W, \mathbb{A}_F)).$ 

- A-parameters ψ are defined as formal sums of φ<sub>i</sub> ⊠ r(b<sub>i</sub>) where φ<sub>i</sub>: cuspidal automorphic representations of GL(n<sub>i</sub>, A<sub>F</sub>), and b<sub>i</sub> ∈ Z<sub>≥1</sub> with parity conditions (Arthur). They are defined without resort to the hypothetical automorphic Langlands group L<sub>F</sub>.
- Also defined:  $S_{\psi}$  and  $S_{\psi}$  equipped with localization maps,  $\forall v$ .
- Given  $\psi$ , can define the summand  $L^2_{\psi}$  of  $L^2_{\text{gen,disc}}$  via "near equivalence classes", using the Satake parameters attached to  $\psi_{\nu}$  for almost all  $\nu$ .
- We say  $\psi$  is *elliptic* if  $\psi = \bigoplus_{i \in I} \phi_i \boxtimes r(b_i)$  where all the  $\phi_i \boxtimes r(b_i)$  are distinct and symplectic.

#### Theorem 1 (Gan–Ichino)

 $L^2_{\text{gen,disc}} = \widehat{\bigoplus}_{\psi:\text{elliptic}} L^2_{\psi}.$ 

The above is proved using  $\Theta\text{-}correspondence$  in stable range. Define

$$\Pi_{\psi} := \left\{ egin{array}{c|c} \pi = (\pi_{v})_{v} & \pi_{v} \in \Pi_{\psi_{v}} ext{ (multi-set!),} \\ ext{ spherical for almost all } v \end{array} 
ight\},$$

and let  $\Pi_\psi(\varepsilon_\psi)$  be the subset given by

$$\prod_{\nu} \langle s_{\nu}, \pi_{\nu} \rangle = \underbrace{\varepsilon_{\psi}^{\operatorname{Art}}(s)}_{\text{same as SO}(2n+1)} \varepsilon \left( \frac{1}{2}, \psi^{s=-1} \big|_{\mathcal{L}_{\mathbf{F}}}, \psi \right), \quad \forall s \in S_{\psi} = S_{\psi}.$$

Main global Theorem (L., conjectured by Gan in ICM 2014) Grosso modo,  $L^2_{\psi} \simeq \bigoplus_{\pi \in \Pi_{\psi}(\boldsymbol{\varepsilon}_{\psi})} \bigotimes_{\nu}' \pi_{\nu}$  for all elliptic  $\psi$ .

#### **Remark.** Levi subgroups of $\mathbf{Sp}(W)$ are of the form

$$\mathbf{M} = \mathbf{Sp}(W^{\flat}) \times \prod_{k=1}^{r} \mathbf{GL}(n_k), \qquad \begin{array}{l} W^{\flat} \subset W : \text{ symp. subspace,} \\ \dim W^{\flat} + 2\sum_k n_k = \dim W. \end{array}$$

One has to formulate and prove these assertions for preimages of  $\mathbf{M}(F)$  (resp.  $\mathbf{M}(\mathbb{A}_F)$ ) in Mp(W) (resp.  $Mp(W, \mathbb{A}_F)$ ).

- 1. The twofold covering does not split over **GL** factors, but this can be handled as in Hanzer–Muić ('10), using some genuine characters made from Weil constants.
- 2. Alternatively, Mp(W) can be enlarged to  $\widetilde{Sp}(W)$  by pushing out via  $\mu_2 \hookrightarrow \mu_8$ . Genuine representation theory is unaffected, but the preimage of  $\mathbf{M}(F)$  becomes  $\widetilde{Sp}(W^{\flat}) \times \prod_{k=1}^{r} \mathsf{GL}(n_k, F)$ .

Both approaches rely on the choice of  $\psi$  and symplectic forms.

## Waldspurger and Gan–Ichino

Consider both the local and global settings.

- When n = 1, these are known to Waldspurger.
- When  $\psi$  is generic (i.e. all  $b_i = 1$ ), the main global theorem is due to Gan–Ichino.
- When n = 2, Gan–Ichino ('21) obtained both main theorems "by hand" with the help of Hanzer–Matić ('10).

These are all based on  $\Theta\mbox{-}correspondence,$  not endoscopy. Compatibilities are shown in [L.]

#### Example: the most degenerate case

Take  $\psi = \mathbf{1} \boxtimes r(2n)$ .

- Locally,  $\Pi_{\psi_{\nu}} = \left\{ \omega_{\psi_{\nu}}^{+}, \omega_{\psi_{\nu}}^{-} \right\}$ ,  $\forall \nu$  (known to Adams).
- Globally,  $L^2_{ij}$  are generated by *elementary*  $\vartheta$ *-series*.

## Proofs: Strategy à la Arthur

Try to imitate Arthur's *endoscopic classification* (Chapters 4 and 7) to prove the local and global theorems altogether. Ingredients:

- Stabilization of trace formula. Done for Mp(W) (L. '21).
- Spectral decomposition of the stable side. Done by Arthur since the endoscopic groups are  $SO(2n' + 1) \times SO(2n'' + 1)$ .
- Local intertwining relation (LIR). DIFFICULT, only known for generic  $\psi$  (due to Ishimoto).

Specifically, Arthur used LIR to prove that the *L*<sup>2</sup>-automorphic spectrum involves only elliptic A-parameters (the "no embedded Hecke eigenvalues" property), for quasi-split classical groups.

It seems difficult to prove LIR directly for  $Mp(W_v)$ .

#### **Shortcut** (+ suggestions from Waldspurger)

Thanks to Gan–Ichino,  $L^2_{gen,disc} = \widehat{\bigoplus}_{\psi:ell,} L^2_{\psi}$  is directly available to us.

- 1. Main global theorem (= decomposition of  $L^2_{\psi}$ ) follows easily from STF for Mp(W,  $\mathbb{A}_F$ ). Though A-packets are not yet available, the global theorem can be formulated as a character relation involving various  $\pi_{\psi_{\nu},\chi_{\nu}}$ .
- 2. The main local theorem is proved by global means via the main global theorem (phrased as above). Data put at the auxiliary places:
  - either from the L-packet inside an A-packet, or
  - suitable co-tempered representations ( $v \nmid \infty$ ).

#### Theorem (F. Chen, '24)

Transfer for Mp(W) commutes with Aubert dual for  $F \supset \mathbb{Q}_p$ .

To get the co-tempered ECR from Luo's ECR via Aubert dual, some sign equality is needed; we globalize carefully and reduce it to SO case (AGIKMS, Ishimoto, Liu–Lou–Shahidi...) via Θ.

## To-do list

- Explicit construction of A-packets when  $F \supset \mathbb{Q}_p$ , after Moeglin, Xu, Atobe..., and multiplicity-one (work in progress by J. Chen).
- Relation to  $\Theta$ -correspondence (Xu's student?).
- Relation to translation functors and cohomological induction when  $F = \mathbb{R}$ , à la Moeglin–Renard; Adams–Johnson packets.
- Explicit construction for  $F = \mathbb{C}$  as predicted by Moeglin–Renard.
- Prove LIR.
- Application to number theory, eg. Ikeda–Yamana lifting from PGL(2) to Mp(2n) for n odd and F totally real.
- Can we use these results to study global root numbers?

## Postscript: affine Hecke algebras

Suppose  $F \supset \mathbb{Q}_p$  and  $\psi$  has conductor  $4\mathbb{O}_F$ . Let  $\mathfrak{G}_{\psi}^{\pm}$  be the Bernstein block  $\ni \mathfrak{u}_{\psi}^{\pm}$ . Let  $\mathfrak{G}^{\pm}$  be the Bernstein block  $\ni \mathfrak{l}_{SO(V^{\pm})}$ .

- Gan–Savin (p > 2) and Takeda–Wood (p = 2) showed  $\mathcal{G}_{\psi}^{\pm} \simeq \mathcal{G}^{\pm}$  by constructing types for  $\mathcal{G}_{\psi}^{\pm}$  and giving an explicit isomorphism between Hecke algebras.
- It is stronger (being a categorical equivalence) and looks more natural than LLC in many aspects. It also preserves unitarity, temperedness and discrete series.
- Mp(W) and SO(V<sup>±</sup>) share the same L-group;  $\mathfrak{G}_{\psi}^{\pm} \simeq \mathfrak{G}^{\pm}$  preserves L-parameters  $\phi$  but not the  $\chi \in \mathfrak{S}_{\phi}^{\vee}$ .

**Natural question 1**: How do the  $\chi$ 's differ under  $\mathcal{G}_{\psi}^{\pm} \simeq \mathcal{G}^{\pm}$ ?

Suppose that  $\pi \in \mathcal{G}_{\psi}^{\pm}$  is irreducible and corresponds to  $\sigma \in \mathcal{G}^{\pm}$ ;

$$\pi = \pi_{\phi,\chi}, \quad \sigma = \sigma_{\phi^{\circ},\chi^{\circ}}$$
 under LLC.

Write  $\phi = \bigoplus_{i \in I} m_i \phi_i$ . Identify  $\mathbb{S}_{\phi}^{\vee}$  with  $\mu_2^{I^+}$  where  $I^+ \subset I$  indexes the symplectic summands in  $\phi$ .

#### Theorem (F. Chen-L. '25)

We have  $\varphi^\circ = \varphi$  (known to GS+TW) and  $\chi^\circ = \chi \nu_\varphi$  , where

$$\mathbf{v}_{\mathbf{\phi}} = (\mathbf{v}_{\mathbf{\phi},i})_{i \in I^+} \in \mathbb{S}_{\mathbf{\phi}}^{\vee}, \quad \mathbf{v}_{\mathbf{\phi},i} = \mathbf{e}\left(\frac{1}{2}, \mathbf{\phi}_i, \psi\right).$$

Modulo [Chen–L. '23], the argument is largely "endoscopic". **Naive question 2**: How about A-packets?

Naive question 3: Other blocks?

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## Thanks for your attention

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