时间: 2016 年 7 月 13 日 10:00-11:40
总分: 100 分

The rings are all commutative with unit element $1 \neq 0$. The field of fractions of a domain R is denoted by $\mathrm{Frac}(R)$.

1. **(20 points)** Suppose R is a normal domain. Show that the polynomial ring $R[X]$ is normal.
Hint. Set $K := \mathrm{Frac}(R)$. Suppose $y \in K(X) = \mathrm{Frac}(R[X])$ is integral over $R[X]$ and $y \neq 0$. First, observe that $y \in K[X]$, say with highest term cX^n. Upon passing to a finitely generated $\mathbb{Z}$-subalgebra of R, we may assume R Noetherian. Integrality implies that $R[X][y]$ is a finitely generated $R[X]$-submodule in $K[X]$. Deduce that the coefficients of all polynomials from $R[X][y] \subset K[X]$ is a finitely generated R-module $\mathcal{C}$, thus so is its submodule $R[c]$. Conclude that $c \in R$ by the normality of R. Now pass to $y - cX^n$, and so forth.

2. **(20 points)** Let R be a domain and write $K := \mathrm{Frac}(R)$.
   (a) Show that if there exists a nonzero homomorphism $\varphi : K \to R$ of R-modules, then R is a field. *Hint*: $t = \varphi(x) \neq 0$ will imply t is divisible by every nonzero element of R since x is.
   (b) Suppose R is not a field. Show that K is flat as an R-module, but not projective. *Hint*: If K is projective over R, there will be an index set I and an embedding $\iota : K \hookrightarrow R^{\oplus I} \subset \prod_{i \in I} R_i$; denote the projections by $p_i : \prod_{j \in I} R_j \to R$ (for all $i \in I$), one of the $\varphi_i := p_i \circ \iota \in \mathrm{Hom}_R(K, R)$ must be nonzero.

3. **(20 points)** Let $\Bbbk$ be a field. Show that $A := \Bbbk[X,Y]/(X) \cap (X,Y)^2$ is not flat over $\Bbbk[Y]$.
Hint. Since $\Bbbk[Y]$ is a PID, it suffices to check whether A is torsion-free or not, as a $\Bbbk[Y]$-module.

4. **(20 points)** Let $I \subset R$ be an ideal whose elements are all nilpotent. Establish the lifting of idempotents as follows.
   (a) Suppose $\bar{a} \in R/I$ satisfies $\bar{a}^2 = \bar{a}$, with preimage $a \in R$. Set $b = 1 - a$. Show that $ab = ba \in I$.
   (b) For a, b as above and $m \geq 1$, put $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$ and

$$e = \sum_{0 \leq k \leq m} \binom{2m}{k} a^k b^{2m-k},$$
$$f = \sum_{m < k \leq 2m} \binom{2m}{k} a^k b^{2m-k}.$$

   Show that $e + f = 1$, $ef = 0$ whenever m is sufficiently large. *Hint*: take $m \gg 0$ so that $(ab)^m = 0$.

(c) Under the assumption $m \gg 0$, deduce that $f^2 = f$ and $f \mapsto \bar{a}$ under the quotient homomorphism.

5. **(20 points)** Let $R = \bigoplus_{n \geq 0} R_n$ be a $\mathbb{Z}_{\geq 0}$-graded ring. Given $d \in \mathbb{Z}_{\geq 1}$, define its d-th Veronese subring as $R_{(d)} := \bigoplus_{k \geq 0} R_{kd}$, which is graded by k.

   (a) Show that R is integral over $R_{(d)}$; deduce $\dim R = \dim R_{(d)}$ under the (realistic) assumption that R and $R_{(d)}$ are both Noetherian. *Hint.* Check integrality for homogeneous elements.

   (b) Recall that an ideal I in a graded ring $\bigoplus_n A_n$ is called homogeneous if it is generated by homogeneous elements; equivalently $I = \bigoplus_n I \cap A_n$. Establish the bijection

$$\{\text{homogeneous primes of } R\} \xrightarrow{1:1} \left\{\text{homogeneous primes of } R_{(d)}\right\}$$
$$\mathfrak{q} \longmapsto \mathfrak{p} := \mathfrak{q} \cap R_{(d)}.$$

   Hint. It is easy to see $\mathfrak{p}$ is homogeneous and prime. Conversely, given $\mathfrak{p}$ we define $\mathfrak{q} = \bigoplus_{k \geq 0} \mathfrak{q}_k$ with $\mathfrak{q}_k := \{r \in R_k : r^d \in \mathfrak{p}\}$. Explain that $\mathfrak{q}$ is a homogeneous prime ideal and show $\mathfrak{q} \leftrightarrow \mathfrak{p}$ are mutually inverse.