Topics in Representation Theory
2020, Peking University

Problem Sheet # 1

Deadline: May 4, 2020

NOTE: You may choose any 3 problems among the following ones.

1. Classify the conjugacy classes in $\mathrm{GL}(2, \mathbb{F}_q)$ into the following types: (i) central, (ii) non-semisimple, (iii) semisimple non-central isotropic, (iv) semisimple anisotropic. Describe these conjugacy classes and count them for each type.

Here we say $g \in \mathrm{GL}(2, \mathbb{F}_q)$ is *semisimple* if g is diagonalizable over $\overline{\mathbb{F}_q}$ as a matrix. A semisimple element is called *isotropic* if it is diagonalizable over $\mathbb{F}_q$, and *anisotropic* otherwise.

Hint. You should obtain $(q-1) + (q-1) + \frac{(q-1)(q-2)}{2} + \frac{q^2-q}{2}$ classes in total.

2. Let F be a field with $\mathrm{char}(F) \neq 2$, assuming for simplicity that $F = \overline{F}$. Let $(W, \langle\cdot,\cdot\rangle)$ be a symplectic F-vector space, which means: W is a finite-dimensional F-vector space and $\langle\cdot,\cdot\rangle : W \times W \to F$ is a non-degenerate bilinear form with $\langle w_1, w_2\rangle = -\langle w_2, w_1\rangle$ for all $w_1, w_2 \in W$. We denote by

$$\mathrm{Sp}(W) := \left\{ g \in \mathrm{GL}(W) : \forall w_1, w_2 \in W,\ \left\langle g w_1, g w_2 \right\rangle = \langle w_1, w_2 \rangle \right\}$$

the corresponding symplectic group.

We say that a vector subspace $V \subset W$ is *totally isotropic* if $\langle\cdot,\cdot\rangle$ is identically zero on $V \times V$. An *isotropic flag* in W is a chain of totally isotropic subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r, \quad r \in \mathbb{Z}_{\geq 0}.$$

If the flag is maximal (i.e. cannot be refined), we call it a *complete flag*.

(i) Sketch a proof that the complete isotropic flags in W form a single $\mathrm{Sp}(W)$-orbit.

(ii) Let $\mathscr{F}$ be a complete isotropic flag. Show that $B := \mathrm{Stab}_{\mathrm{Sp}(W)}(\mathscr{F})$ is a parabolic subgroup of $\mathrm{Sp}(W)$, i.e. $\mathrm{Sp}(W)/B$ is a projective variety.

(iii) Show that B is also a solvable group.

(iv) Set $n := \frac{1}{2}\dim_F W$ and let $\Omega_n := \{\pm 1\}^n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ acts on $\{\pm 1\}^n$ by permutation; Ω_n can also be the subgroup of $\mathfrak{S}_{2n}$ acting on $\{\pm 1, \dots, \pm n\}$ consisting of permutations commuting with $i \mapsto -i$.

Try to use the case of $\mathrm{GL}(2n)$ to define the invariant $\mathrm{inv}(\mathcal{F}, \mathcal{G}) \in \Omega_n$ associated to any two complete isotropic flags $\mathcal{F}, \mathcal{G}$ so that $\mathrm{inv}(\mathcal{F}, \mathcal{G}) = \mathrm{inv}(\mathcal{F}', \mathcal{G}')$ if and only if $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ lie in the same $\mathrm{Sp}(W)$-orbit.

You may make free use of the properties of symplectic vector spaces in your textbooks, eg. the existence of symplectic bases, etc.

☞ **Hint**. Each complete isotropic flag $\mathcal{F} : 0 = V_0 \subsetneq V_1 \subsetneq \cdots$ extends to a complete flag in W

$$\tilde{\mathcal{F}} : 0 = V_0 \subsetneq \cdots \subsetneq \underbrace{V_n = V_n^\perp}_{\text{Lagrangian}} \subsetneq V_{n-1}^\perp \subsetneq \cdots \subsetneq 0^\perp = W.$$

Hence $\mathrm{Stab}_{\mathrm{Sp}(W)}(\mathcal{F}) = \mathrm{Stab}_{\mathrm{GL}(W)}(\tilde{\mathcal{F}}) \cap \mathrm{Sp}(W)$. For any pair $\mathcal{F}, \mathcal{G}$ of complete isotropic flags, we obtain $\mathrm{inv}(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}) \in \mathfrak{S}_{2n}$. Show that it must lie in Ω_n.

3. Let (W, S) be a Coxeter system. We say $w \in W$ is a *reflection* if it is conjugate to some $s \in S$. Denote by $\mathrm{fs}(W)$ the set of reflections in W, and by $2^{\mathrm{fs}(W)}$ its power-set. Prove the following statements.

(i) There exists a unique map $N : W \to 2^{\mathrm{fs}(W)}$ such that $N(s) = \{s\}$ for all $s \in S$, and $N(xy) = N(y) \Delta y^{-1} N(x) y$ for all $x, y \in W$ where $a \Delta b := (a \cup b) \setminus (a \cap b)$ is the symmetric difference of sets.

(ii) If $w = s_1 \cdots s_n$ is a reduced expression of $w \in W$ (with $s_i \in S$ for all i), and $s \in S$ satisfies $\ell(ws) < \ell(w) = n$, then there exists some $1 \leq i \leq n$ such that

$$ws = s_1 \cdots \widehat{s_i} \cdots s_n$$

where $\widehat{\cdots}$ means the term to be removed. This is called the *Exchange Condition* for Coxeter systems. Also show that there is an equivalent "left" counterpart concerning sw.

(iii) Any two reduced expressions of $w \in W$ can be transformed to each other by applying the *braid relations* consecutively, namely: for all $s, t \in S$ with $m(s, t) \neq \infty$,

$$sts\cdots = tst\cdots, \quad \text{with } m(s,t) \text{ terms on both sides.}$$

☞ **Hint**. This nontrivial result is part of Theorem 2.1.2 in Digne and Michel, *Representations of finite groups of Lie type*, Second Edition (2020). Read it!

4. Use the Exchange Condition in the previous result to show that if (W, S) is a Coxeter system and W is finite, then there exists a unique element $w_0 \in W$ of maximal length, and it satisfies $w_0^2 = 1$.

☞ **Hint**. This is part of Proposition 2.1.5 in Digne and Michel, *Representations of finite groups of Lie type*, Second Edition (2020). Read it!

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Problem Sheet # 2

Deadline: June 17, 2020

NOTE: You may choose any 2 problems among the following ones.

Conventions In the following problems, we fix a finite field $\mathbb{F}_q$ and a reductive group $\mathbf{G}$ over $\mathbb{F}_q$, with the usual convention $G := \mathbf{G}(\mathbb{F}_q)$, etc. We fix a prime number $\ell \nmid q$ to define the virtual characters $R_{\mathbf{T}}^{\theta} = R_{\mathbf{T}}^{\theta,\mathbf{G}}$ of Deligne–Lusztig valued in $\overline{\mathbb{Q}_\ell}$. Here $\mathbf{T} \subset \mathbf{G}$ is a maximal torus over $\mathbb{F}_q$, and $\theta : T \to \overline{\mathbb{Q}_\ell}^\times$ is a homomorphism.

You may make free use the properties of ℓ-adic cohomologies mentioned in class.

1. Let $g = su$ be a Jordan decomposition in the linear algebraic group $\mathbf{G}$, where s is semisimple and u is unipotent. Assume that $g \in G$. Show that $s, u \in G$ and $g = su$ is also the Jordan decomposition in the finite group G.

☞ **Hint**. The uniqueness of Jordan decomposition implies that s, u are Galois-invariant, hence rational. As for the second assertion, it suffices to show that semisimple (resp. unipotent) elements have order prime to p (resp. powers of p); for this purpose, one may pass to any finite extension $\mathbb{F}_{q^d}$.

2. Let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U} \subset \mathbf{G}$ be a parabolic subgroup over $\mathbb{F}_q$, and let $\mathbf{T} \subset \mathbf{L}$ be a maximal torus over $\mathbb{F}_q$. Show that $R_{\mathbf{T}}^{\theta,\mathbf{G}} = i_P^G \left(R_{\mathbf{T}}^{\theta,\mathbf{L}} \right)$.

☞ **Hint**. This is Proposition 8.2 in P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields* (1976).

3. Show that if a maximal torus $\mathbf{T} \subset \mathbf{G}$ over $\mathbb{F}_q$ is not contained in any proper parabolic subgroup $\mathbf{P} = \mathbf{LU} \subsetneq \mathbf{G}$ over $\mathbb{F}_q$, then for any $\theta : T \to \overline{\mathbb{Q}_\ell}^\times$ in general position, the irreducible representation $\pm R_{\mathbf{T}}^{\theta}$ of G is cuspidal.

☞ **Hint**. This is Theorem 8.3 in P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields* (1976). You may use the previous problem together with the decomposition of the regular representation of L in terms of Deligne–Lusztig virtual characters.

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