Modular Forms and Number Theory
2021, Peking University

Problem Sheet # 2

Deadline: June 21th, 2021

NOTE: Please choose any 2 problems among the following ones. You can use the results covered in class.

1. Let $k \in \mathbb{Z}$, $k \neq 0$ and let $\gamma \in \mathrm{SL}(2, \mathbb{R})$ be an elliptic transformation, i.e. γ has exactly one fixed point in $\mathscr{H}$. Let $f : \mathscr{H} \to \mathbb{C}$ be a holomorphic function. Show that if $f \mid_k \gamma = f$ and γ has infinite order, then $f = 0$.

☞ **Hint**. First reduce to the case $\gamma(i) = i$. Next, reduce to the case $k \in 2\mathbb{Z}$ so that the problem can be interpreted by differential forms (or their $\otimes$-powers). Translate the problem onto the unit disc.

2. For every $\tau \in \mathscr{H}$, put $\Lambda_\tau := \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$. The endomorphism ring of the complex torus $\mathbb{C}/\Lambda_\tau$ is denoted as $\mathrm{End}(\mathbb{C}/\Lambda_\tau)$, which is a subring of $\mathbb{C}$. Show that $\mathrm{End}(\mathbb{C}/\Lambda_\tau) \supsetneq \mathbb{Z}$ if and only if τ is a quadratic irrational in $\mathscr{H}$, i.e. there exist $A, B, C \in \mathbb{Z}$ such that $A \neq 0$ and $A\tau^2 + B\tau + C = 0$.

☞ **Hint**. First, show that $\mathrm{End}(\mathbb{C}/\Lambda_\tau) \supsetneq \mathbb{Z}$ if and only if $\gamma\tau = \tau$ for some $\gamma \in \mathrm{GL}(2, \mathbb{Q})^+$ which is not a scalar. Show that the quadratic irrationals in $\mathscr{H}$ are precisely the fixed points of elements of $\mathrm{GL}(2, \mathbb{Q})^+$. See §8.6 of the textbook.

3. Let $k > 2$ be an even integer. Show that the Eisenstein series E_k is orthogonal to $S_k(\mathrm{SL}(2, \mathbb{Z}))$ with respect to the Petersson inner product.

☞ **Hint**. Write $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ and $\Gamma_\infty := \mathrm{Stab}_\Gamma(\infty)$. Argue that, for all $f \in S_k(\mathrm{SL}(2, \mathbb{Z}))$,

$$\int_{\Gamma\backslash\mathscr{H}} f(\tau) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{j(\gamma, \tau)^{-k}} \mathrm{Im}(\tau)^k \, \mathrm{d}\mu(\tau) = \int_{\Gamma_\infty \backslash \mathscr{H}} f(\tau) \mathrm{Im}(\tau)^{k-2} \, \mathrm{d}x \, \mathrm{d}y$$

where $\mathrm{d}\mu(\tau) = y^{-2}\mathrm{d}x\mathrm{d}y$ (with $\tau = x + iy$) is the hyperbolic measure on $\mathscr{H}$. Find a fundamental domain for $\mathscr{H}$ under Γ_∞-action, and observe that $\int_0^1 f(x + iy) \, \mathrm{d}x = 0$ for each $y \in \mathbb{R}$.

4. For each even integer $k > 2$, use the Eisenstein series G_k to define

$$\mathscr{G}_k := \frac{(k-1)!}{2(2\pi i)^k} \cdot G_k \ \in M_k(\mathrm{SL}(2,\mathbb{Z}))$$

so that $a_n\left(\mathscr{G}_k\right) = \sigma_{k-1}(n)$ for all $n \geq 1$. Show that $\mathscr{G}_k$ is a normalized Hecke eigenform satisfying $T_p\mathscr{G}_k = \left(1 + p^{k-1}\right)\mathscr{G}_k$ for every prime number p.

☞ **Hint.** Compare $a_n\left(T_p\mathscr{G}_k\right)$ and $a_n\left(\mathscr{G}_k\right)$. The case $n = 0$ is straightforward. As for the case $n \geq 1$, one has to determine $\left(1 + p^{k-1}\right)\sigma_{k-1}(n) = \sigma_{k-1}(p)\sigma_{k-1}(n)$ in terms of $\sigma_{k-1}(pn)$ and $\sigma_{k-1}(n/p)$ (when $p \mid n$).

5. (Reading homework: real-analytic Eisenstein series) Let $\tau = x + iy \in \mathscr{H}$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define

$$E(\tau, s) := \frac{\pi^{-s}\Gamma(s)}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|m\tau + n|^{2s}}.$$

Show that

(i) the sum converges absolutely when $\mathrm{Re}(s) > 1$, and

$$E(\tau, s) = \frac{1}{2} \cdot \pi^{-s}\Gamma(s)\zeta(2s) \sum_{\gamma} \mathrm{Im}(\gamma\tau)^s$$

where $\mathrm{Re}(s) > 1$ and γ ranges over $\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ & 1 \end{smallmatrix}\right)\backslash\mathrm{SL}(2,\mathbb{Z})$.

(ii) $E(\gamma\tau, s) = E(\tau, s)$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$;

(iii) $E(\tau, s)$ extends to a meromorphic function in $s \in \mathbb{C}$ and satisfies

$$E(\tau, s) = E(\tau, 1 - s).$$

Note that $E(\tau, s)$ is only real-analytic in τ.

☞ **Hint.** For (iii), see either

- D. Bump, *Automorphic forms and representations*, Theorem 1.6.1 / Exercise 1.6.2, or
- F. Diamond and J. Shurman, *A first course in modular forms* (GTM 228), §4.10.

The second reference treats a more generalized setting with level N, and takes the sum of $y^s(m\tau + n)^{-k}|m\tau + n|^{-2s}$ instead; taking $s = 0$ recovers the holomorphic Eisenstein series.

Last update: June 10, 2021