# Exercises in Commutative Ring Theory

The Enhanced Program for Graduate Study, BICMR

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# Preface

The present document comprises the homework for the course *Commutative Ring Theory* (Spring 2019) for the 11th Enhanced Program for Graduate Study organized by the Beijing International Center for Mathematical Research. They are divided into 11 problem sets, corresponding roughly to the materials in the lecture notes of the first-named author. Due to time constraints, they cover only a very limited terrain of Commutative Algebra. Nevertheless, we hope this will be of some use to the readers.

Most of the exercises herein are collected from existing resources such as the textbooks by Atiyah–Macdonald, Eisenbud, Matsumura, etc., or the websites MathOverflow and Stacks Project. The authors claim no originality of the exercises.

We are deeply grateful to the students of this course, coming from all over mainland China, for their active participation and valuable feedback.

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## General conventions

- Unless otherwise specified, the rings are assumed to be commutative with unit 1. The ring of integers, rational, real and complex numbers are denoted by the usual symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.
- An expression $A := B$ means that $A$ is defined to be $B$. Injections and surjections are indicated by $\hookrightarrow$ and $\twoheadrightarrow$, respectively. The difference of sets is denoted as $A \setminus B$, etc.
- The group of invertible elements in a ring $R$ is denoted by $R^\times$. The localization of $R$ with respect to a multiplicative subset $S$ is denoted by $R[S^{-1}]$, and the localization with respect to a prime ideal $\mathfrak{p}$ by $R_\mathfrak{p}$. The space of prime ideals (resp. maximal ideals) of $R$ is denoted by $\mathrm{Spec}(R)$ (resp. $\mathrm{MaxSpec}(R)$) and we let $V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supset \mathfrak{a}\}$ for any ideal $\mathfrak{a}$. The ideal generated by elements $a, b, \ldots$ is denoted by $(a, b, \ldots)$.
- The field of fractions of an integral domain $R$ is denoted by $\mathrm{Frac}(R)$.
- The support of an $R$-module $M$ is denoted by $\mathrm{Supp}(M)$, and the set of associated prime by $\mathrm{Ass}(M)$. The annihilator ideals are denoted by $\mathrm{ann}(\cdots)$.
- For a ring $R$, the $R$-algebra of polynomials (resp. formal power series) is denoted by $R[X, Y, \ldots]$ (resp. $R[\![X, Y]\!]$) where $X, Y, \ldots$ stand for the indeterminates.

# Problem set 1

# 28 February

You may select 5 of the following problems as your homework.

*Ring Theory Revisited*

1. Let $x$ be a nilpotent element of ring A. Show that $1 + x$ is a unit of $A$ . Deduce that the sum of a nilpotent element and a unit is a unit.
2. Let $A$ be a ring and let $A[x]$ be the ring of polynomials in an an indeterminate $x$, with coefficients in $A$ . Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that:

   (1) $f$ is a unit in $A[x] \iff a_0$ is a unit in $A$ and $a_1, \dots, a_n$ are nilpotent.

   (**Hint**: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of $f$, prove by induction on $r$ that $a_n^{r+1}b_{m-r} = 0$. Hence show that $a_n$ is nilpotent, and then use Ex.1.)

   (2) $f$ is nilpotent $\iff$ $a_0, a_1, \dots, a_n$ are nilpotent.

   (3) $f$ is a zero-divisor $\iff$ there exists $a \neq 0$ in $A$ such that $af = 0$.

   (**Hint**: Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree $m$ such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$.)

   (4) $f$ is said to be primitive if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then $fg$ is primitive $\iff$ $f$ and $g$ are primitive.
3. Prove that a local ring contains no element $e \neq 0, 1$, such that $e^2 = e$.

*Zariski Topology*

4. For each $f \in A$, let $X_f$ denote the complement of $V(f)$ in $X = \mathrm{Spec}(A)$. The sets $X_f$ are open. Show that they form a basis of open sets for the Zariski topology, and that

   (1) $X_f \cap X_g = X_{fg}$

   (2) $X_f = \emptyset \iff f$ is nilpotent.

   (3) $X_f = X \iff f$ is a unit.

*Prime Avoidance*

5. Here are two examples that show how the prime avoidance cannot be improved.

(1) Show that if $k = \mathbb{Z}/(2)$ then the ideal $(x, y) \subset k[x, y]/(x, y)^2$ is the union of 3 properly smaller ideals.

(2) Let $k$ be any field. In the ring $k[x, y]/\left(xy, y^2\right)$, consider the ideals $I_1 = (x)$, $I_2 = (y)$, and $J = \left(x^2, y\right)$. Show that the homogeneous elements of $J$ are contained in $I_1 \cup I_2$, but that $J \not\subseteq I_1$ and $J \not\subseteq I_2$. Note that one of the $I_j$ is prime.

*Localization of rings and modules*

6. Let $f : A \to B$ be a homomorphism of rings and let $S$ be a multiplicatively closed subset of $A$. Let $T = f(S)$. Show that $B[S^{-1}]$ and $B[T^{-1}]$ are isomorphic as $A[S^{-1}]$-modules.

7. Let $M$ be an $A$-module. Then the following are equivalent:

   (1) $M = 0$.

   (2) $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of $A$.

   (3) $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of $A$.

*Nakayama's lemma*

8. Let $A$ be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical $\mathfrak{R}$ of $A$, let $M$ be an $A$-module and $N$ a finitely generated $A$-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then $u$ is surjective.

# Problem set 2

# 7 March

You may select 3 of the following problems as your homework.

*Radicals*

1. A ring $A$ is such that every ideal not contained in the nilradical (i.e. $\sqrt{0}$) contains a non-zero idempotent (that is, an element $e$ such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of $A$ are equal.

*Noetherian and Artinian rings*

2. In a Noetherian ring $A$, show that every ideal $\mathfrak{a}$ contains a power of its radical.
3. Let $M$ be an $A$-module and let $N_1, N_2$ be submodules of $M$. If $M/N_1$ and $M/N_2$ are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

*Support of a module*

4. (1) Let $A$ be a local ring, $M$ and $N$ finitely generated $A$ -modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

   (**Hint**: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$, and similarly definition for $N_k$. Then use Nakayama's lemma, and note that $M_k, N_k$ are vector spaces.)

   (2) Let $A$ be a ring, $M, N$ are finitely generated $A$-module. Prove that $\mathrm{Supp}\,(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$

# Problem set 3

# 14 March

You may select 4 of the following problems as your homework.

*Support, Associated primes and Primary decompositions*

1. Show that the support of the $\mathbb{Z}$-module $M := \bigoplus_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$ is not a closed subset of $\mathrm{Spec}(\mathbb{Z})$. Show that its closure equals $V(\mathrm{ann}_{\mathbb{Z}}(M))$.
2. Let $p$ be a prime number. Show that the support of the $\mathbb{Z}$-module $\prod_{a=1}^{\infty} \mathbb{Z}/p^a\mathbb{Z}$ is not equal to the closure of $\bigcup_{a=1}^{\infty} \mathrm{Supp}\left(\mathbb{Z}/p^a\mathbb{Z}\right)$.
3. Let $M$ be an $R$-module, which is "graded" in the following sense: we are given decompositions of abelian groups $R = \bigoplus_{a=0}^{\infty} R_a$ and $M = \bigoplus_{b=0}^{\infty} M_b$, such that $1 \in R_0$ and for all $a, b \in \mathbb{Z}_{\geq 0}$,

$$R_a \cdot R_b \subset R_{a+b}, \quad R_a \cdot M_b \subset M_{a+b}.$$

   Suppose that $m \in M$ and $\mathfrak{p} := \mathrm{ann}_R(m)$ is a prime ideal of $R$.

   (1) Show that $\mathfrak{p} = \bigoplus_{a\geq 0}(\mathfrak{p} \cap R_a)$ .We say $\mathfrak{p}$ is a "homogeneous ideal".

   (2) Show that there exist some $b \geq 0$ and $m' \in M_b$ such that $\mathfrak{p} = \mathrm{ann}_R(m')$ .Thus $\mathfrak{p}$ is actually the annihilator of some "homogeneous element".

4. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

   Regard the $\mathbb{C}$-vector space $\mathbb{C}^3$ as a $\mathbb{C}[T]$ -module via $\left(\sum_{j=0}^{m} a_j T^j\right) v := \sum_{j=0}^{m} a_j \left(A^j v\right)$. What are the associated prime ideals of this $\mathbb{C}[T]$ -module?

5. Try to give a primary decomposition of the ideal $I = \left(x^3y, xy^4\right)$ of the ring $R = k[x, y]$.

   (**Hint**:If $I$ is a monomial ideal with a generator $ab$, where $a$ and $b$ are coprime, say $I = (ab) + J$, then $I = ((a) + J) \cap ((b) + J)$. Applying this recursively gives a primary decomposition.)

# Problem set 4

# 21 March

You may select 4 of the following problems as your homework.

*Integral dependence, Nullstellensatz*

1. Prove that the integral closure of $R := \mathbb{C}[X, Y]/(Y^2 - X^2 - X^3)$ in $\mathrm{Frac}(R)$ equals $\mathbb{C}[t]$ with $t := \bar{Y}/\bar{X}$, where $\bar{X}, \bar{Y}$ denote the images of $X, Y$ in $R$.

   (**Hint**: First, show that the homomorphism $\xi : \mathbb{C}[X, Y] \to \mathbb{C}[T]$ given by $X \mapsto T^2 - 1$ and $Y \mapsto T^3 - T$ has kernel equal to $(Y^2 - X^2 - X^3)$. Secondly, show that $T$ is in the field of fractions of $R \simeq \mathrm{image}(\xi) \subset \mathbb{C}(T)$, integral over $R$, and argue that $\mathbb{C}[T]$ is the required integral closure. Try to motivate the choice of $\xi$.)

2. Consider a Noetherian ring $R$ with $K := \mathrm{Frac}(R)$. Show that $x \in K$ is integral over $R$ if and only if there exists $u \in R$ such that $u \neq 0$ and $ux^n \in R$ for all $n$.

   (**Hint**: When this condition holds, $R[x]$ is an $R$-submodule of $u^{-1}R$.)

3. Let $R = \mathbb{Q}[X_1, X_2, \ldots]$ (finite or infinitely many variables). Show that $\mathrm{nil}(R) = \mathrm{rad}(R) = \{0\}$.

   (**Hint**: To show $\mathrm{rad}(R) = \{0\}$, produce a surjective homomorphism $\varphi_f : R \to \mathbb{Q}$ such that $\varphi_f(f) \neq 0$, for any given nonzero $f \in R$.)

4. Let $R = \mathbb{Q}[X_1, X_2, \ldots]$ (infinitely many variables). Show that $R$ is not a Jacobson ring.

   (**Hint**: Exhibit a surjective homomorphism $\psi : R \to R'$ such that $R'$ is an integral domain with unique maximal ideal $\neq \{0\}$. Such $R'$ can be obtained by localizing $\mathbb{Q}[X]$ at some prime ideal.)

5. (1) Let $A$ be a subring of an integral domain $B$ , and let $C$ be the integral closure of $A$ in $B$. Let $f, g$ be monic polynomials in $B[x]$ such that $fg \in C[x]$ . Then $f, g$ are in $C[x]$.

   (**Hint**: Take a field containing $B$ in which the polynomials $f, g$ split into linear factors, prove that the coefficients of $f$ and $g$ are integral over $C$.)

   (2) Prove the same result without assuming that $B$ (or $A$) is an integral domain.

6. Let $f : A \to B$ be an injective map, and $B$ integral over $A$. Assume that neither $A$ nor $B$ have zero divisors.

   (1) Show that if $A$ is a field, then so is $B$ .

   (2) Deduce that a field $k$ is algebraically closed (i.e., every polynomial has a root) if and only if for every finite field extension $k \subset k'$ i.e., $k'$ is f.d. as a $k$-vector space, we have $k = k'$.

   (3) Show that if $B$ is a field, then so is $A$ .

# Problem set 5

# 28 March

*Flatness*

1. For a field $k$ , show that $k[\![t]\!][Y,Z]/(YZ-t)$ is flat over $k[\![t]\!]$.
2. Let $N', N, N''$ be $A$-modules, and $0 \to N' \to N \to N'' \to 0$ be an exact sequence, with $N''$ flat. Prove that $N'$ is flat $\iff$ $N$ is flat.

   (**Hint**: Use the Tor exact sequence, and note that for any $A$-module $M$, $\mathrm{Tor}_n(M, N'') = \mathrm{Tor}_n(N'', M) = 0$. Then use 5-lemma.)
3. A ring $A$ is absolutely flat if every $A$ -module is flat. Prove that the following are equivalent:

   (1) $A$ is absolutely flat.

   (2) Every principal ideal is idempotent.

   (3) Every finitely generated ideal is a direct summand of $A$.

   (**Hint**: For (1) $\Longrightarrow$ (2), let $x \in A$, consider the following diagram:

$$\begin{tikzcd} (x) \otimes A \arrow[r] \arrow[d] & A/(x) \arrow[d] \\ A \arrow[r] & A/(x) \end{tikzcd}$$

   For (2) $\Longrightarrow$ (3), prove that every finitely generated ideal is principal.)
4. Prove the following properties of absolutely flat:

   (1) Every homomorphic image of an absolutely flat ring is absolutely flat.

   (2) If a local ring is absolutely flat, then it is a field.

   (3) If a ring $A$ is absolutely flat, then every non-unit in $A$ is a zero-divisor.

# Problem set 6

# 4 April

*Going-up and Going-down*

1. Let $f : A \to B$ be an integral homomorphism of rings, i.e. $B$ is integral over its subring $f(A)$. Show that $f^{\#} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed mapping, i.e. that it maps closed sets to closed sets.

   (**Hint**: This is a geometrical equivalent of the first part of Krull–Cohen–Seidenberg theorem. )

2. Let $A \subset B$ be an extension of rings, making $B$ integral over $A$, and let $\mathfrak{p}$ be a prime ideal of $A$. Suppose there is a unique prime ideal $\mathfrak{q}$ of $B$ with $\mathfrak{q} \cap A = \mathfrak{p}$. Show that

   (a) $\mathfrak{q}B_{\mathfrak{p}}$ is the unique maximal ideal of $B_{\mathfrak{p}} := B[(A \setminus \mathfrak{p})^{-1}]$;

   (b) $B_{\mathfrak{q}} = B_{\mathfrak{p}}$;

   (c) $B_{\mathfrak{q}}$ is integral over $A_{\mathfrak{p}}$.

   (**Hint**: Observe that $B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$. For (a), show that $\mathfrak{q}B_{\mathfrak{p}}$ is the unique prime of $B_{\mathfrak{p}}$ over $\mathfrak{p}A_{\mathfrak{p}}$, then apply the local case of Krull–Cohen–Seidenberg Theorem. For (b) and (c), show that every $y \in B \setminus \mathfrak{q}$ becomes invertible in $B_{\mathfrak{p}}$ by using the integrality over $A$.)

3. Let the integral extension $A \subset B$ and the prime ideal $\mathfrak{p}$ be as above. Suppose that $A$ is a domain and $\mathfrak{q}, \mathfrak{q}'$ are distinct prime ideals of $B$, both mapping to $\mathfrak{p}$ under $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Show that $B_{\mathfrak{q}}$ is not integral over $A_{\mathfrak{p}}$.

   (**Hint**: Take $y \in \mathfrak{q}' \setminus \mathfrak{q}$. Claim: $y^{-1} \in B_{\mathfrak{q}}$ is not integral over $A_{\mathfrak{p}}$. Otherwise we would have

   $$a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1} y = -1$$

   for some $a_0, \ldots a_{n-1} \in A_{\mathfrak{p}}$ with $a_0 \neq 0$. Clean denominators of $a_0, \ldots, a_{n-1}$ to derive a contradiction.)

# Problem set 7

# 11 April

*Graded rings and modules, Filtrations*

1. Many basic operations on ideals, when applied to homogeneous ideals in $\mathbb{Z}$-graded rings, lead to homogeneous ideals. Let $I$ be a homogeneous ideal in a $\mathbb{Z}$-graded ring $R$. Show that:

   (1) The radical of $I$ is homogeneous, that is, the radical of $I$ is generated by all the homogeneous elements $f$ such that $f^n \in I$ for some $n$.

   (2) If $I$ and $J$ are homogeneous ideals of $R$, then $(I : J) := \{f \in R | fJ \subset I\}$ is a homogeneous ideal.

   (3) Suppose that for all $f, g$ homogeneous elements of $R$ such that $fg \in I$, one of $f$ and $g$ is in $I$. Show that $I$ is prime.

2. Suppose $R$ is a $\mathbb{Z}$-graded ring and $0 \neq f \in R_1$.

   (1) Show that $R[f^{-1}]$ is again a $\mathbb{Z}$-graded ring.

   (2) Let $S = R[f^{-1}]_0$, show that $S \cong R/(f-1)$, and $R[f^{-1}] \cong S[x, x^{-1}]$, where $x$ is a new variable.

3. Show that if $R$ is a graded ring with no nonzero homogeneous prime ideals, then $R_0$ is a field and either $R = R_0$ or $R = R_0[x, x^{-1}]$.

4. Taking the associated graded ring can also simplify some features of the structure of $R$. For example, let $k$ be a field, and let $R = k[x_1, \dots, x_r] \subset R_1 = k[\![x_1, \dots, x_r]\!]$ be the rings of polynomials in $r$ variables and formal power series in $r$ variables over $k$, and write $I = (x_1, \dots, x_r)$ for the ideal generated by the variables in either ring. Show that $\operatorname{gr}_I R = \operatorname{gr}_I R_1$.

# Problem set 8

# 18 April

You may select 3 of the following problems as your homework.

*Completions*

1. Let $A$ be a local ring, $\mathrm{m}$ its maximal ideal. Assume that $A$ is $\mathrm{m}$ -adically complete. For any polynomial $f(x) \in A[x]$, let $f(x) \in (A/\mathrm{m})[x]$ denote its reduction mod. $\mathrm{m}$. Prove Hensel's lemma: if $f(x)$ is monic of degree $n$ and if there exist coprime monic polynomials $\overline{g}(x), \overline{h}(x) \in (A/\mathrm{m})[x]$ of degrees $r, n-r$ with $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$, then we can lift $\overline{g}(x), \overline{h}(x)$ back to monic polynomials $g(x), h(x) \in A[x]$such that $f(x) = g(x)h(x)$.

   (**Hint**:Assume inductively that we have constructed $g_k(x), h_k(x) \in A[x]$ such that

   $$g_k(x)h_k(x) - f(x) \in \mathfrak{m}^k A[x].$$

   Then use the fact that since $\overline{g}(x)$ and $\overline{h}(x)$ are coprime we can find $\overline{a}_p(x), \overline{b}_p(x)$, of degrees $\leqslant n-r, r$ respectively, such that $x^p = \overline{a}_p(x)\overline{g}_k(x) + \overline{b}_p(x)\overline{h}_k(x)$, where $p$ is any integer such that $1 \leqslant p \leqslant n$. Finally, use the completeness of $A$ to show that the sequences $g_k(x), h_k(x)$converge to the required $g(x), h(x)$.)

2. (1) With the notation of Exercise 1, deduce from Hensel's lemma that if $\overline{f}(x)$ has a simple root $\alpha \in A/\mathrm{m}$, then $f(x)$ has a simple root $a \in A$ such that $\alpha = a \bmod \mathrm{m}$.

   (2) Show that 2 is a square in the ring of 7 -adic integers.

   (3) Let $f(x,y) \in k[x,y]$, where $k$ is a field, and assume that $f(0,y)$ has $y = a_0$ as a simple root. Prove that there exists a formal power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $f(x, y(x)) = 0$. This gives the "analytic branch" of the curve $f = 0$ through the point $(0, a_0)$.

3. Let $A$ be a Noetherian ring, $\mathfrak{a}$ an ideal in $A$, and $\hat{A}$ the $\mathfrak{a}$-adic completion. For any $x \in A$, let $\hat{x}$ be the image of $x$ in $\hat{A}$ . Show that $x$ not a zero-divisor in $A$ implies $\hat{x}$ not a zero-divisor in $\hat{A}$. Does this imply that if $A$ is an integral domain then $\hat{A}$ is an integral domain?

   (**Hint**: For a counter-example, consider $R = k[x,y]$ and

   $$\mathfrak{m} = (x,y), \quad A = k[x,y]/(y^2 - x^2 - x^3).$$

Let $f \in k[\![x, y]\!]$ be such that $f^2 = 1 + x$, such a power series can be obtained$f = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \in k[\![x]\!] \subset k[\![x, y]\!])$)

4. Let $\Bbbk$ be a field and consider the quotient of infinite polynomial ring

$$R := \frac{\Bbbk[X, Z, Y_1, Y_2, Y_3, \ldots]}{\left(X - ZY_1, X - Z^2Y_2, X - Z^3Y_3, \ldots\right)}.$$

Denote by $\overline{Z}$ the image of $Z$ in $R$. Show that the ideal $I := (\overline{Z})$ of $R$ satisfies $\bigcap_{n \geq 1} I^n \neq \{0\}$. Why is this consistent with Krull's intersection theorem?

# Problem set 9

# 25 April

The materials below are taken from the Stacks Project.

*Mittag-Leffler systems and Completions for non-Noetherian rings*

1. Consider an inverse system of *sets* $\cdots \leftarrow A_n \overset{\varphi_{n+1}}{\leftarrow} A_{n+1} \leftarrow \cdots$ (where $n = 1, 2, \ldots$). For each $j > i$, let $\varphi_{j,i} : A_j \to A_i$ be the composition of $\varphi_j, \ldots, \varphi_i$. We say that *Mittag-Leffler* conditions holds for $(A_n, \varphi_n)_{n \geq 1}$ if for each $i$, we have
$$\varphi_{k,i}(A_k) = \varphi_{j,i}(A_j) \quad \text{whenever } j, k \gg i.$$
Show that if $(A_n, \varphi_n)_n$ is Mittag-Leffler and $A_n \neq \emptyset$ for each $n$, then the limit
$$\varprojlim_n A_n := \left\{ (a_n)_n \in \prod_n A_n : \forall n,\ \varphi_{n+1}(a_{n+1}) = a_n \right\}$$
is nonempty as well.

2. Suppose we are given an inverse system of short exact sequences of abelian groups, i.e. a commutative diagram
$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[u] & B_n \arrow[r, "g_n"] \arrow[u] & C_n \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"'] \arrow[u, "\varphi_{n+1}"] & B_{n+1} \arrow[r, "g_{n+1}"'] \arrow[u, "\psi_{n+1}"] & C_{n+1} \arrow[r] \arrow[u, "\xi_{n+1}"'] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$
with exact rows, where $n = 1, 2, \ldots$. Show that if $(A_n, \varphi_n)_n$ is Mittag-Leffler, then
$$0 \to \varprojlim A_n \xrightarrow{\lim f_n} \varprojlim B_n \xrightarrow{\lim g_n} \varprojlim C_n \to 0$$
is exact. You only have to show the surjectivity of $\lim g_n$.

3. Let $R$ be a ring (not necessarily Noetherian), $I$ be a proper ideal, and $\varphi : M \to N$ be a homomorphism of $R$-modules. Prove the following statements.

   (a) If $M/IM \to N/IN$ is surjective, then so is $\hat{\varphi} : \hat{M} \to \hat{N}$. Here $\hat{M} = \varprojlim_{n \geq 1} M/I^n M$ stands for the $I$-adic completion.

   (**Hint:** First, show $M/I^n M \to N/I^n N$ is surjective. Next, set $K_n = \mathrm{Ker}[M \to N/I^n N]$ to get exact sequences

   $$0 \to K_n/I^n M \to M/I^n M \to N/I^n N \to 0,$$

   then try to establish the surjectivity of $K_{n+1}/I^{n+1}M \to K_n/I^n M$ in order to apply Mittag-Leffler.)

   (b) If $0 \to K \to M \to N \to 0$ is an exact sequence of $R$-modules and $N$ is flat, then $0 \to \hat{K} \to \hat{M} \to \hat{N} \to 0$ is exact.

   (**Hint:** Tensor by $R/I^n$ to obtain an inverse system of short exact sequences, and apply Mittag-Leffler.)

   (c) If $M$ is finitely generated, then the natural homomorphism $M \otimes_R \hat{R} \to \hat{M}$ given by $m \otimes (r_n)_{n=1}^{\infty} \mapsto (r_n m)_{n=1}^{\infty}$ is surjective.

   (**Hint:** Show that if $R^{\oplus N} \to M$ is a surjective homomorphism, then so is $\widehat{R^{\oplus N}} \to \hat{M}$.)

4. Suppose $I$ is finitely generated. Let $M$ be an $R$-module. Prove that

$$I^n \hat{M} = \mathrm{Ker}[\hat{M} \to M/I^n M] = \widehat{I^n M}$$

for all $n \in \mathbb{Z}_{\geq 1}$, and $\hat{M}$ is $I$-adically complete as an $R$-module.

(**Hint:** Fix $n$ and take generators $f_1, \ldots, f_r$ of $I^n$. This yields a surjective homomorphism of $R$-modules $(f_1, \ldots, f_r) : M^{\oplus r} \to I^n M \subset M$. Pass to completions and show that

$$(f_1, \ldots, f_r) : \hat{M}^{\oplus r} \to \widehat{I^n M} = \varprojlim_{m \geq n} \frac{I^n M}{I^m M} \simeq \mathrm{Ker}[\hat{M} \to M/I^n M] \subset \hat{M},$$

which is surjective by the previous exercise. Identify the image of $(f_1, \ldots, f_r) : \hat{M}^{\oplus r} \to \hat{M}$ to infer that $\hat{M}/I^n \hat{M} \simeq M/I^n M$.)

# Problem set 10

# 9 May

*Hilbert series*

1. Let $\Bbbk$ be a field and $R = \Bbbk[X_0, \dots, X_n]$, graded by total degree. Consider the graded $R$-module $S = R/(f)$ where $f$ is a homogeneous polynomial of total degree $d \geq 1$. Show that when $m \geq d$,
$$\chi(S, m) := \dim_{\Bbbk} S_m = \binom{m+n}{n} - \binom{m+n-d}{n}.$$

2. Let $R = \bigoplus_n R_n$ be a $\mathbb{Z}_{\geq 0}$-graded ring, finitely generated over $R_0$. Assume $R_0$ is Artinian (for example, a field) and let $M = \bigoplus_n M_n$ be a finitely generated $\mathbb{Z}_{\geq 0}$-graded $R$-module. Define the *Hilbert series* in the variable $T$ as
$$H_M(T) := \sum_{m \geq 0} \chi(M, m) T^m \quad \in \mathbb{Z}[\![T]\!]$$
where $\chi(M, m)$ denotes the length of the $R_0$-module $M_m$, as usual. In what follows, graded means graded by $\mathbb{Z}_{\geq 0}$.
   (a) Show that if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of graded $R$-modules, then $H_M(T) = H_{M'}(T) + H_{M''}(T)$.
   (b) Relate $H_M(T)$ and $H_{M(k)}(T)$ for arbitrary $k \in \mathbb{Z}$, where $M(k)_d := M_{d+k}$.
   (c) Suppose that $R$ is generated as an $R_0$-algebra by homogeneous elements $x_1, \dots, x_n$ with $d_i := \deg x_i \geq 1$. Show that there exists $Q \in \mathbb{Q}[T]$ such that
$$H_M(T) = \frac{Q(T)}{(1 - t^{d_1}) \cdots (1 - t^{a_n})}$$
as elements of $\mathbb{Q}[\![T]\!]$.

   (**Hint:** you may imitate the arguments for the quasi-polynomiality of $\chi(M, n)$.)
   (d) What can be said about the $\mathbb{Z}_{\geq 0}^m$-graded case, for general $m$?

# Problem set 11

# 16 May

*Dimension theory*

1. Let $\mathbb{Z}_3$ be the 3-adic completion of the ring $\mathbb{Z}$, so that $\mathbb{Z} \hookrightarrow \mathbb{Z}_3$ naturally. Evaluate $1 + 3 + 3^2 + 3^3 + \cdots$ in $\mathbb{Z}_3$.

2. Let $R$ be a Noetherian local ring. Suppose that there exists a principal prime ideal $\mathfrak{p}$ in $R$ such that $\mathrm{ht}(\mathfrak{p}) \geq 1$. Prove that $R$ is an integral domain.
(**Hint**: Below is one possible approach. Suppose $\mathfrak{p} = (x)$ for some $x \in R$. Let $\mathfrak{q} \subset \mathfrak{p}$ be a minimal prime in $R$. Argue that (i) $x \notin \mathfrak{q}$, (ii) $\mathfrak{q} = x\mathfrak{q}$, and finally (iii) $\mathfrak{q} = \{0\}$.)

3. Let $\Bbbk$ be a field and $R = \Bbbk[\![X]\!] \times \Bbbk[\![X]\!]$. Prove that $R$ is a Noetherian semi-local ring, $R$ contains a principal prime ideal of height 1, but $R$ is not an integral domain.
(**Hint**: It is known that $\Bbbk[\![X]\!]$ is Noetherian local with maximal ideal $(X)$. Argue that the ideals in $\Bbbk[\![X]\!] \times \Bbbk[\![X]\!]$ take the form $I \times J$ where $I, J$ are ideals in $\Bbbk[\![X]\!]$. Show that $(X) \times \Bbbk[\![X]\!]$ and $\Bbbk[\![X]\!] \times (X)$ are the only maximal ideals, and both are of height 1.)

# Final exam of Commutative Ring Theory

The Enhanced Program for Graduate Study, BICMR / PKU
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The rings are assumed to be commutative and nonzero, but NOT necessarily Noetherian. We say that a ring homomorphism $\rho : A \to B$ is *faithful* if for all nonzero $A$-module $M$, we have $M \underset{A}{\otimes} B \neq \{0\}$.

1. **(20 points)** Show that the polynomial algebra $\mathbb{Q}[X_1, X_2, X_3, \ldots]$ in infinitely many variables has infinite Krull dimension.

   **Solution:** Consider the prime chain $(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots$.

2. **(15 points)** Let $\rho : A \to B$ be a faithful homomorphism of rings and let $u : M \to N$ be a homomorphism of $A$-modules. Prove that if $u \otimes \mathrm{id}_B : M \underset{A}{\otimes} B \to N \underset{A}{\otimes} B$ is surjective, then $u : M \to N$ is surjective as well.

   **Solution:** Recall that $\mathrm{coker}(u \otimes \mathrm{id}_B) = \mathrm{coker}(u) \underset{A}{\otimes} B$.

In what follows, we fix a ring $A$ and an element $f \in A$ which is NOT a zero-divisor. Denote by $\widehat{A}$ the $(f)$-adic completion of $A$. We identify $f$ with its image in $\widehat{A}$ and write $A_f := A[f^{-1}]$, $\widehat{A}_f := \widehat{A}[f^{-1}]$ for the localizations with respect to the multiplicative subsets generated by $f$. More generally, we write $M_f := M[f^{-1}]$ for any module $M$ over $A$ or $\widehat{A}$. Note that $M \xrightarrow{\sim} M_f$ if $f : M \to M$ is an automorphism.

3. **(20 points)** Show that $f$ is not a zero divisor for $\widehat{A}$.

   **Solution:** Let $\hat{a} = (a_n)_{n \geq 1} \in \varprojlim_{n \geq 1} A/(f^n)$. Then $f\hat{a} = 0$ is equivalent to $a_{n+1} \in (f^n)$ for all $n \geq 1$. This implies $a_n = 0$ for all $n \geq 1$, since $a_n = a_{n+1} \bmod f^n$.

4. **(15 points)** Let $M$ be an $A$-module such that every element $x \in M$ is annihilated by some power of $f$. Show that

   (i) when $f^n M = \{0\}$ for some $n \geq 1$, the canonical homomorphism $M \to M \underset{A}{\otimes} \widehat{A}$ given by $x \mapsto x \otimes 1$ is bijective;

   (ii) same as (i), but without assuming $f^n M = \{0\}$;

   (iii) the "diagonal" homomorphism $\rho : A \to A_f \times \widehat{A}$ is faithful.

   You may use the easy fact that $\widehat{A}/f^n\widehat{A} \simeq A/(f^n)$ (an earlier homework).

   **Solution:** Consider (i). We have

$$M \simeq M \underset{A}{\otimes} \frac{A}{(f^n)} \xrightarrow{\sim} M \underset{A}{\otimes} \frac{\widehat{A}}{f^n\widehat{A}} \simeq M \underset{A}{\otimes} \widehat{A},$$

   whose composite is exactly $M \to M \underset{A}{\otimes} \widehat{A}$. For (ii), write

$$M = \bigcup_{n \geq 1} \ker\left[f^n : M \to M\right]$$

and commute $\varinjlim$ past $\otimes$. As for (iii), note that our condition amounts to $M_f = \{0\}$.

5. **(20 points)** Let $L$ be an $\widehat{A}$-module for which $f$ is not a zero-divisor, thus $L$ can be viewed as a submodule of $L_f$. Prove that

   (i) $\mathrm{Tor}_1^A\left(\widehat{A}, L/f^n L\right)$ is zero for all $n \geq 1$, in particular $f$ is not a zero divisor for $L \underset{A}{\otimes} \widehat{A}$;

   (ii) $\mathrm{Tor}_1^A\left(\widehat{A}, L_f/L\right)$ is zero.

   You may make free use of the long exact sequence for Tor:

$$\cdots \to \mathrm{Tor}_{i+1}^A(X, Y'') \to \mathrm{Tor}_i^A(X, Y') \to \mathrm{Tor}_i^A(X, Y) \to \mathrm{Tor}_i^A(X, Y'') \to \cdots$$

   where $0 \to Y' \to Y \to Y'' \to 0$ is exact, as well as the fact that Tor commutes with direct limits and direct sums in each variable.

   **Solution:** In view of

$$L_f/L \simeq \varinjlim_{n \geq 1} \frac{1}{f^n} L/L, \qquad \frac{1}{f^n} L/L \simeq L/f^n L,$$

   it suffices to prove (i). The case $L = \widehat{A}$ amounts to the result that $f$ is not a zero divisor in $\widehat{A}$, since $\widehat{A}/f^n\widehat{A} \simeq A/(f^n)$. For general $L$, one chooses a presentation of the $A/(f^n)$-module $L/f^n L$: there exists a set $I$ and an exact sequence of $A$-modules

$$0 \to M \to \left(A/(f^n)\right)^{\oplus I} \to \frac{L}{f^n L} \to 0.$$

   Then, by the previous result,

$$\begin{aligned} \mathrm{Tor}_1^A\left(\widehat{A}, L/f^n L\right) &\simeq \ker\left[M \underset{A}{\otimes} \widehat{A} \to \left(A/(f^n)\right)^{\oplus I} \underset{A}{\otimes} \widehat{A}\right] \\ &\simeq \ker\left[M \to \left(A/(f^n)\right)^{\oplus I}\right] = \{0\}. \end{aligned}$$

6. **(10 points)** Below is a weak version of the celebrated Beauville–Laszlo theorem. Consider the following data

   - an $A_f$-module $K$,
   - an $\widehat{A}$-module $L$ for which $f$ is not a zero-divisor,
   - an isomorphism $\varphi : K \otimes_A \widehat{A} \xrightarrow{\sim} L_f$ of $\widehat{A}_f$-modules.

   Show that there exists an $A$-module $M$, for which $f$ is not a zero-divisor, together with isomorphisms

$$\alpha : M_f \xrightarrow{\sim} K, \qquad \beta : M \underset{A}{\otimes} \widehat{A} \xrightarrow{\sim} L,$$

   such that $\varphi$ equals the composite of

$$K \underset{A}{\otimes} \widehat{A} \xrightarrow{\alpha^{-1} \otimes \mathrm{id}} M_f \underset{A}{\otimes} \widehat{A} \xrightarrow{\beta[f^{-1}]} L_f.$$

**Solution:** Given $(K, L, \varphi)$, let us begin by showing the surjectivity of the composed homomorphism

$$\overline{\varphi} : K \to K \underset{A}{\otimes} \widehat{A} \xrightarrow{\varphi} L_f \to L_f/L.$$

Since $A \to A_f \times \widehat{A}$ is faithful and $(L_f/L)_f = 0$, this reduces to the surjectivity of the $\widehat{A}$-linear map

$$K \underset{A}{\otimes} \widehat{A} \to (L_f/L) \underset{A}{\otimes} \widehat{A} \simeq L_f/L,$$

which agrees with $\overline{\varphi}$ when pulled-back to $K$, thus it equals $K \underset{A}{\otimes} \widehat{A} \xrightarrow[\sim]{\varphi} L_f \twoheadrightarrow L_f/L$. Hence $\overline{\varphi}$ is surjective.

Next, put $M := \ker(\overline{\varphi})$ into the short exact sequence

$$0 \to M \xrightarrow{i} K \xrightarrow{\overline{\varphi}} L_f/L \to 0.$$

Thus $\alpha := i[f^{-1}] : M_f \to K_f \simeq K$ is an isomorphism since $K$ is already an $A_f$-module. Also, $f$ is not a zero-divisor for $M$. Using $\mathrm{Tor}_1^A(\widehat{A}, L_f/L) = 0$, we conclude that the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \underset{A}{\otimes} \widehat{A} \arrow[r, "i \otimes \mathrm{id}"] & K \underset{A}{\otimes} \widehat{A} \arrow[r] \arrow[d, "\varphi", "\wr"'] & L_f/L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & L_f \arrow[r] & L_f/L \arrow[r] & 0
\end{tikzcd}$$

has exact rows. We obtain $\beta := \varphi \circ (i \otimes \mathrm{id}) : M \underset{A}{\otimes} \widehat{A} \xrightarrow{\sim} L$. Since $(L_f/L)_f = 0$, by localizing the diagram above with respect to $f$ we obtain

$$\begin{tikzcd}
M_f \underset{A}{\otimes} \widehat{A} \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[d, equal] & K \underset{A}{\otimes} \widehat{A} \arrow[d, equal] \\
(M \underset{A}{\otimes} \widehat{A})_f \arrow[r] \arrow[d, "{\beta[f^{-1}]}"'] & (K \underset{A}{\otimes} \widehat{A})_f \arrow[d, "\varphi"] \\
L_f \arrow[r, "\mathrm{id}"'] & L_f
\end{tikzcd}$$

in which all arrows are invertible. It follows that $\beta[f^{-1}] \circ (\alpha^{-1} \otimes \mathrm{id}) = \varphi$.

**Note.** Our approach to Beauville–Laszlo theorem follows the original paper

A. Beauville, Y. Laszlo, *Un lemme de descente.* (French. Abridged English version). C. R. Acad. Sci., Paris, Sér. I 320, No. 3, 335-340 (1995).

A more general version can be found on Stacks Project, 15.81.